A DYNAMIC OPTIMIZATION TECHNIQUE FOR PREDICTING MUSCLE FORCES IN THE SWING PHASE OF GAIT

D. T. DAVY and M. L. AUDU
Orthopaedic Engineering Laboratory, Department of Mechanical and Aerospace Engineering, Case Western Reserve University, Cleveland, OH 44106, U.S.A.

Abstract—The muscle force sharing problem was solved for the swing phase of gait using a dynamic optimization algorithm. For comparison purposes the problem was also solved using a typical static optimization algorithm. The objective function for the dynamic optimization algorithm was a combination of the tracking error and the metabolic energy consumption. The latter quantity was taken to be the sum of the total work done by the muscles and the enthalpy change during the contraction. The objective function for the static optimization problem was the sum of the cubes of the muscle stresses. To solve the problem using the static approach, the inverse dynamics problem was first solved in order to determine the resultant joint torques required to generate the given hip, knee and ankle trajectories. To this effect the angular velocities and accelerations were obtained by numerical differentiation using a low-pass digital filter. The dynamic optimization problem was solved using the Fletcher-Reeves conjugate gradient algorithm, and the static optimization problem was solved using the Gradient-restoration algorithm. The results show influence of internal muscle dynamics on muscle control histories vis a vis muscle forces. They also illustrate the strong sensitivity of the results to the differentiation procedure used in the static optimization approach.

INTRODUCTION

One useful aim of the analysis of musculoskeletal motion is the quantification of muscle actions during the observed motion history. The muscle forces play a major role in determining joint contact forces and stresses in the bones. The muscle actions also reflect the underlying neural control processes which are of particular interest in understanding and dealing with neuromuscular disabilities.

Since invasive measurements of kinetic and kinematic data cannot be used, analysis of musculoskeletal motion has typically involved modeling the system as an actuator-driven linkage and making external measurements of motion histories (Bresler and Frankel, 1950). With appropriate data the models can predict resultant actions between linkages necessary to produce the motion (the inverse dynamics problem; Chau and Riu, 1973). The well recognized difficulty in actually determining the muscle forces is the so-called mechanical redundancy problem (Crowninshield, 1978; Hardt, 1978; Patriarco et al., 1981). Although some characteristics of muscle behavior have been incorporated into a few analyses (Hardt, 1978; Pedotti et al., 1978), the optimization approaches by the workers cited above have incorporated the assumption that muscle actions at any instant are independent of actions at all other points in time. In the present paper we refer to this approach as static optimization after Hardt (1978), since no excitation and contraction dynamics of the muscles are included.

Optimal control methods, which allow for the incorporation of muscle dynamics, have been used in motion synthesis (Chow and Jacobson, 1971) and optimal motion problems (Hatze, 1976; Hatze, 1981). In this paper we present an initial study of the application of optimal control analysis, or dynamic optimization in contrast to static optimization, to solving the muscle force distribution problem. We consider lower limb motion during the swing phase of gait. The model incorporates nine muscle groups and a mixed optimality criterion involving both a tracking error and an energy consumption term. The dynamic muscle model incorporates a single control input and several features of excitation/contraction dynamics (Audu and Davy, 1985). Results are presented in the form of control histories and muscle force histories during the motion. For comparison purposes, a static optimization solution is also found based on a previously proposed optimality criterion (Crowninshield and Brand, 1981).

Received January 1985; in revised form May 1986.
THE DYNAMIC OPTIMIZATION PROBLEM

Statement of the problem

The system under consideration is the lower limb consisting of four rigid bodies—the pelvis, the thigh, the shank and the foot. This system is depicted diagrammatically in Fig. 1. All movement of the system is restricted to the sagittal plane and the swing phase of the limb alone is considered. The orientation and \( x, y \) position of the pelvis are specified functions of time. Therefore the resulting system (Fig. 1a) has three degrees-of-freedom—the rotations about the hip, knee and ankle joints, all modeled as hinge joints. The muscle actuator system (Fig. 1b) is modeled in terms of nine muscle groups—iliopsoas, hamstrings, rectus femoris, gastrocnemius, short head of biceps femoris, vasti, tibialis anterior, soleus and gluteus maximus.

The resulting dynamic optimization problem can be stated as follows: given the hip, knee and ankle trajectories \( \theta_1(t), \theta_2(t), \theta_3(t) \) and the pelvic trajectory \( x(t), y(t) \) (pelvic orientation taken as constant), for the swing phase of gait; find the muscle controls \( u(t) \) (and hence the muscle forces \( F_M(t) \)) that will generate the given trajectories while minimizing the total muscular effort expended in the process. By total muscular effort is meant the metabolic energy consumed as evidenced by the enthalpy change and the mechanical work done by the muscles. Viewed from an optimal control point of view, this problem is analogous to a nonlinear tracking problem with a limitation on the amount of energy consumed in the process. An appropriate objective function for the problem will take the form

\[
I = \int_{t_0}^{t_f} \{ \tau(t)^T R \tau(t) + \int_{t_0}^{t_f} \{ \tau(t)^T H \tau(t) + G E(t) \} dt \} \tag{1}
\]

where

\[
z(t) = x(t)^d - x(t).
\]

In these equations \( x(t) \) is the generated trajectory, \( x^d(t) \) is the desired trajectory and \( B, G \) and \( H \) are positive definite (or semidefinite) weighting matrices. \( E(t) \) is a measure of the total energy involved in the process.

The general form of the dynamic optimization problem considers the minimization of the functional

\[
I = [g(x, p)]_{t_f} + \int_{t_0}^{t_f} L(x, u, p, t) dt \tag{2}
\]

with respect to the state vector \( x(t) \), the control vector \( u(t) \), and the parameter vector \( p \) which satisfy the vector differential constraint

\[
\dot{x} = f(x, u, p, t) \tag{3}
\]

the nondifferential constraint

\[
S(x, u, p, t) = 0 \tag{4}
\]

and the boundary conditions

\[
x(t_0) = x_0 \text{ given} \tag{5}
\]

\[
[\phi(x, p)]_{t_f} = 0. \tag{6}
\]

In the above equations, the functions \( L \) and \( g \) are scalar, the function \( f \) is an \( n \)-vector, the function \( S \) is an \( r \)-vector, and the function \( \phi \) is a \( q \)-vector. The independent variable is the time \( t \) (a scalar), and the dependent variables are the state \( x \) (an \( n \)-vector), the control \( u \) (an \( m \)-vector) and the parameter \( p \) (a \( d \)-vector).

At the initial time \( t = t_0 \), the \( n \) scalar relations (5) are specified. At the final time \( t = t_f \), the \( q \) scalar relations (6) are specified.
Equations (3) are the dynamics equations which in this case consist of the dynamics equations describing the muscular subsystem and those describing the skeletal subsystem. The equations used to describe the dynamics of the skeletal subsystem are the equations of motion for a system of connected rigid bodies which are derived using D'Alembert's principle. The main equations used to model the muscular subsystem will be discussed below.

The muscle model

The muscle model used in this study is a lumped model consisting of four elements (Fig. 2). This model was chosen on the basis of previous work examining the influence of muscle model complexity on musculoskeletal dynamics (Audu and Davy, 1985). The muscle model is described in detail elsewhere (Audu and Davy, 1985; Audu, 1985); its essential features will be described here for the sake of completeness. The four model elements are the contractile element (CE), the series elastic element (SE) and the parallel elements (PE) and (DE). The CE models the active part of the muscle. The dynamics of this element consists of the excitation dynamics and the contraction dynamics. The excitation dynamics of the muscle is taken to be related to the release of calcium ions from the sarcoplasm of the muscle and its subsequent binding to the contraction molecular structures (Hatze, 1980). The contraction dynamics is defined by the force-velocity and the force-length relationships of the muscle contractile tissue. The SE and the PE elements are modeled as nonlinear springs. The damping element DE is modeled as a linear damper. The equations describing these features are given below.

Following our earlier work (Audu and Davy, 1985) we describe the excitation dynamics of the CE in terms of a first order time dependent relation. The active state \( q \), \( 0 \leq q \leq 1 \), is taken to be a saturating function

\[
q(g) = 1 - b_1 \exp(-b_2 g).
\]

The parameter \( g, 0 < g < 1 \), which can be regarded as a normalized measure of bound Ca ions (Taylor, 1969) is described by the first order differential equation

\[
\dot{g}(s) = b_3 (b_4 s - g).
\]

The parameter \( s \) in equation (8) is the single control input to the muscle, which is considered to be a normalized stimulation frequency. The constants \( b_1, b_2, b_3, b_4 \) in equations (7) and (8) are muscle specific parameters describing saturation levels and stimulus response rates (Audu and Davy, 1985) and are listed in Table 1.

Let the normalized length of the contractile element be given by

\[
L_c = \frac{L_c - L_c}{L_c - L_c}
\]

where \( L_c \) and \( L_c \) are the instantaneous length and optimum length respectively of the contractile element. The latter quantity is that length at which the contractile element produces maximum isometric force.

![Fig. 2. Muscle model consisting of a contractile element CE and a series elastic element SE, in parallel with an elastic element PE and a damping element DE. Muscle length L is sum of CE length L_c and SE length L_c.](image)

### Table 1. Parameters in state equations for dynamic optimization (SI units)

<table>
<thead>
<tr>
<th>Muscle group</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_1 )</td>
<td>9.6</td>
<td>9.6</td>
<td>11.6</td>
<td>9.6</td>
<td>6.9</td>
<td>11.6</td>
<td>6.9</td>
<td>6.9</td>
<td>9.6</td>
</tr>
<tr>
<td>( b_2 )</td>
<td>0.11</td>
<td>0.12</td>
<td>0.11</td>
<td>0.13</td>
<td>0.13</td>
<td>0.14</td>
<td>0.12</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>( b_3 )</td>
<td>0</td>
<td>0</td>
<td>0.07</td>
<td>0</td>
<td>0.04</td>
<td>0.14</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( b_4 )</td>
<td>3800</td>
<td>5800</td>
<td>1900</td>
<td>3600</td>
<td>1450</td>
<td>4400</td>
<td>850</td>
<td>4100</td>
<td>6900</td>
</tr>
<tr>
<td>( b_5 )</td>
<td>0.23</td>
<td>0.252</td>
<td>0.35</td>
<td>0.38</td>
<td>0.247</td>
<td>0.088</td>
<td>0.198</td>
<td>0.312</td>
<td>0.24</td>
</tr>
<tr>
<td>( b_6 )</td>
<td>0.099</td>
<td>0.09</td>
<td>0.108</td>
<td>0.15</td>
<td>0.034</td>
<td>0.147</td>
<td>0.085</td>
<td>0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>( b_7 )</td>
<td>77.3</td>
<td>85.1</td>
<td>70.9</td>
<td>51.1</td>
<td>87.9</td>
<td>52.1</td>
<td>90.1</td>
<td>85.7</td>
<td>95.4</td>
</tr>
<tr>
<td>( b_8 )</td>
<td>1049</td>
<td>1601</td>
<td>525</td>
<td>904</td>
<td>400</td>
<td>1214</td>
<td>235</td>
<td>1312</td>
<td>1905</td>
</tr>
<tr>
<td>( b_9 )</td>
<td>11.5</td>
<td>11.5</td>
<td>15.5</td>
<td>11.5</td>
<td>9.5</td>
<td>15.5</td>
<td>9.5</td>
<td>9.5</td>
<td>11.5</td>
</tr>
<tr>
<td>( b_{10} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( b_{11} )</td>
<td>0.575</td>
<td>0.575</td>
<td>0.7</td>
<td>0.625</td>
<td>0.425</td>
<td>0.7</td>
<td>0.425</td>
<td>0.425</td>
<td>0.575</td>
</tr>
<tr>
<td>( b_{12} )</td>
<td>0.152</td>
<td>0.152</td>
<td>0.185</td>
<td>0.165</td>
<td>0.112</td>
<td>0.185</td>
<td>0.112</td>
<td>0.112</td>
<td>0.152</td>
</tr>
<tr>
<td>( b_{13} )</td>
<td>0.25</td>
<td>0.25</td>
<td>0.23</td>
<td>0.21</td>
<td>0.22</td>
<td>0.25</td>
<td>0.24</td>
<td>0.23</td>
<td>0.24</td>
</tr>
<tr>
<td>( a_1 )</td>
<td>0.78</td>
<td>0.55</td>
<td>0.77</td>
<td>0.85</td>
<td>0.71</td>
<td>0.90</td>
<td>0.17</td>
<td>0.81</td>
<td>0.89</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>0.83</td>
<td>0.83</td>
<td>0.87</td>
<td>0.83</td>
<td>0.75</td>
<td>0.87</td>
<td>0.75</td>
<td>0.75</td>
<td>0.83</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>55.5</td>
<td>55.5</td>
<td>70</td>
<td>55.5</td>
<td>13.5</td>
<td>70</td>
<td>13.5</td>
<td>13.5</td>
<td>55.5</td>
</tr>
<tr>
<td>( c )</td>
<td>275</td>
<td>275</td>
<td>300</td>
<td>275</td>
<td>200</td>
<td>300</td>
<td>200</td>
<td>200</td>
<td>275</td>
</tr>
<tr>
<td>( k_5 )</td>
<td>5.85</td>
<td>4.1</td>
<td>5.4</td>
<td>8.25</td>
<td>1.6</td>
<td>6.8</td>
<td>1.3</td>
<td>6.5</td>
<td>9.1</td>
</tr>
<tr>
<td>( k_6 )</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>15</td>
</tr>
</tbody>
</table>
The normalized force–velocity relationship for the CE can be written as

\[
\hat{Q}_t = \begin{cases} 
- b_{14}(F - F_{SE})/(F_{SE} + b_{14}F) & \text{for } F_{SE} \leq F \\
- b_{16}(F - F_{SE})/(1.33F - F_{SE}) & \text{for } F_{SE} > F
\end{cases}
\]  

for \( F_{SE} \leq F \).

The first portion of equation (10) is the Hill force–velocity relationship which describes the contraction dynamics of the CE during contracting velocities (Audu and Davy, 1985). The second portion models the phenomenon of muscle yielding which occurs during lengthening velocities. In this equation \( F \) is the isometric force of the muscle which is obtained by modulating the maximum isometric force \( P \) by the length–tension relationship of the CE, \( k(Q_o) \), and the active state function, \( q(s) \), i.e.

\[
F = \hat{F}k(Q_o)q(s).
\]

The length–tension relationship \( k(Q_o) \) is modeled as an exponential after Hatze (1980) and is defined as

\[
k(Q_o) = \exp(- (Q_o - L)^2/b_3).
\]

The force in the SE is also defined by an exponential (Hatze, 1980)

\[
F_{SE} = b_6(\exp(1.531\delta) - 1).
\]

In equation (13), \( \delta \) is a normalized length given by

\[
\delta = (L_s - L_{inf})(L_s - L_{inf})
\]

where \( L_s, L_r \) and \( L_{inf} \) are the instantaneous length, optimum length and rest length of the SE respectively.

The various constants in equations (7–14) were chosen by methods described elsewhere (Audu, 1985; Audu and Davy, 1985) and are listed for the nine muscle groups in Table 1.

The torques exerted by the passive joint structures (ligaments) acting across the joints are modeled using the equation (Audu and Davy, 1985)

\[
M_j = k_j \exp(-k_2(x_j - \theta_{2j})) - k_j \exp(-k_4(\theta_{1j} - x_j))
\]

for \( j = 1, 2, 3 \).

The passive parallel elastic elements (PE) for the muscles are modeled using similar equations as for the passive joint structures. Since in general these elements exert force in only one direction a single exponential was used. The equation for the force takes the general form

\[
F_{PE} = \hat{k}_5(\exp(\hat{k}_6(L - L_0)) - 1) + cL
\]

where \( L \) is the total length of the muscle, \( L_0 \) is the rest length and \( \hat{L} \) is the muscle velocity. The parameters \( c, k_5 \) and \( \hat{k}_6 \) in equation (17) were chosen for the various muscle groups as described by Audu and Davy (1985) and are given in Table 1.

### Link kinematics

Following the works of Chow and Jacobson (1971) and of Mena et al. (1981), the pelvis is assumed to progress forward (in the x-direction) with a constant velocity during the swing phase. The vertical motion (in the y-direction) is modeled as a sinusoidal movement of the general form

\[
y(t) = h \sin(4\pi t + b)
\]

where \( h \) is the amplitude of the motion and \( b \) is the phase. The amplitude \( h \) is of the order of 1 in. (2.54 cm) (Capozzo and Pedotti, 1975). A nominal value for \( b \) has been shown by Mena et al. (1971) to be 0.1 rad for the swing phase of normal gait.

The specified joint trajectories \( \theta_1(t), \theta_2(t) \) and \( \theta_3(t) \) which are used in both the dynamic and static optimization approaches were obtained from Mann et al. (1975).

### Anthropometric parameters

In the model a subject height of 1.6 m and mass of 80 kg were assumed. The lengths, masses and moments of inertia of the limbs were calculated using the ratios given by Winter (1979). Origins and insertions of the muscles were estimated using the data of Brand et al. (1982). Details of parameter evaluation and numerical values are given in Audu (1985).

### The rate of energy consumption

The rate of metabolic energy consumption in a given activity can be shown to consist of five terms (Hatze and Buys, 1977; Mommaerts, 1969)

\[
E = A + M + S + W + D
\]

where \( A \) is the muscle activation heat rate, \( M \) is the muscle maintenance heat rate, \( S \) is the muscle shortening heat rate, \( W \) is the muscle mechanical work rate,
and $D$ is the rate of energy dissipation in the passive structures. The empirical relationships describing each of these quantities were derived as follows.

(a) The activation heat rate. Based on the experimental works of Gibbs and Gibson (1972), Hatze and Buys (1977) modeled this phenomenon as an exponential function of the stimulation frequency $f$

$$A = kfH/H_m f_{\text{max}}$$  \hspace{1cm} (20)

where

$$H = 1 - \exp(- (e_1 + e_2/f))$$

$$H_m = 1 - \exp(- (e_1 + e_2/f_{\text{max}})).$$

In these equations $k$, $e_1$, and $e_2$ are constants and $f_{\text{max}}$ is the maximum stimulation frequency.

Defining $f/f_{\text{max}} = s$, equation (20) can be written as

$$A = ks H/H_m$$  \hspace{1cm} (21)

where

$$H = 1 - \exp(- (e_1 + e_2/f_{\text{max}} s))$$

$$H_m = 1 - \exp(- (e_1 + e_2/f_{\text{max}})).$$

Numerical values for the constants $e_1$ and $e_2$ have been given by Hatze and Buys (1977) as 18.2 and 0.25 respectively. A plot of $A$ vs $s$ readily demonstrates that for most practical purposes $A$ can be modeled as a linear function of normalized frequency $s$ except at very low frequencies (Audu, 1985). Since at such frequencies the value of $A$ is relatively small, only a small error (on the order of 2% for normalized frequencies greater than 0.1) is introduced by using the following linear approximation.

Let $H_A$ be the activation heat rate per unit mass. Then an appropriate model for the activation heat is

$$H_A = a_1 f + b_1$$

where $a_1$ and $b_1$ are constants. When $f = f_{\text{max}}$, $H_A = f_0$ and when $f = 0$, $H_A \approx 0$, therefore

$$A = W_m f_0 s$$  \hspace{1cm} (22)

where $f_0$ is the muscle specific activation heat rate in W/kg and $W_m$ is the muscle mass in kg.

(b) The maintenance heat rate. In their experiments Gibbs and Gibson (1972) classified the maintenance heat rate into a tension-dependent heat rate and a tension-independent heat rate. The tension-independent heat rate is essentially the activation heat rate. Gibbs and Gibson (1972) have shown that the tension-dependent heat rate is a linear function of the isometric force. In equation (11) the isometric force is given as a function of the muscle length and the active state of the muscle.

An appropriate form for the tension-dependent heat rate is therefore

$$H_1 = a_2 F = a_2 \bar{F} k(Q_0) q(s)$$  \hspace{1cm} (23)

where $H_1$ is the muscle maintenance heat rate per unit mass, $a_2$ is a constant, $\bar{F}$ is the isometric force of the muscle at the optimum length of the contractile element $\bar{L}$, and at maximum stimulation frequency, $k(Q_0)$ is the muscle length–tension relationship defined in equation (12), $q(s)$ is the active state of the muscle defined in equation (7) and $Q_0$ is the normalized length of the contractile element.

Identifying the constant $a_2 \bar{F}$ with the muscle specific maintenance heat rate $f_m$, we get

$$M = W_m f_m k(Q_0) q(s).$$  \hspace{1cm} (24)

The muscle masses $W_m$ were calculated using the relationship

$$W_m = \rho_m A_m L_0$$  \hspace{1cm} (25)

where $\rho_m$ is the density of muscle tissue taken as 1000 kg m$^{-3}$ (Hatze, 1980), $A_m$ is the muscle cross-sectional area and $L_0$ is the rest length of the muscle.

(c) The shortening heat rate. The extra heat produced as a consequence of the shortening of the muscle is called the shortening heat rate. Following similar reasoning as in Hill (1953), this quantity can be modeled by the relation

$$S = aV.$$  \hspace{1cm} (26)

The constant of proportionality $a$ is the same as that defined in the Hill force–velocity relationship and is given by

$$a = b_1 F.$$  \hspace{1cm} (27)

In these equations $V$ is the isometric contraction velocity defined by the relation $V = - L \dot{Q}_0 \bar{L}$, is the isometric length of the contractile element, $F$ is the maximum isometric force of the muscle and $b_1$ is a constant.

Substituting for $F$ from equation (11) $S$ takes the final form

$$S = - b_1 \ddot{L} \dot{Q}_0 \bar{F} k(Q_0) q(s).$$  \hspace{1cm} (28)

(d) The mechanical work rate. The mechanical work rate is the product of the muscle force and the muscle contraction velocity; i.e.

$$W = - L \dot{Q}_0 \bar{F} s$$  \hspace{1cm} (29)

where $s$ is the force in the series elastic element.

(e) Rate of dissipation in the passive structures. This quantity is given by the product of the dissipative force in the damping element of the muscle and the muscle contraction velocity; i.e.

$$D = c \bar{L}^2$$  \hspace{1cm} (30)

where $\bar{L}$ is the total muscle velocity and $c$ is the dissipation constant.

Summary of dynamic equations

The dynamic optimization problem consists of determining the control histories for the muscle groups which minimize the objective function, equation (1), subject to the state equations representing the limb and muscle dynamics. The complete set of state equations for the dynamic optimization problem are given in Appendix A. The complete set of equations includes the six link equations of motion involving limb angles.
and angular velocities; eighteen muscle dynamics equations involving muscle active state and contractile element lengths, and one equation due to the explicit appearance of $t$ as a parameter in the dynamic equations. Thus, the dimension of the state vector is 25.

The dimension of the control vector is nine, corresponding to the normalized frequencies of stimulation for the nine muscle groups.

THE STATIC OPTIMIZATION PROBLEM

Statement of the problem

In the static optimization approach to this same problem, muscle dynamics are excluded. The muscles are considered to be instantaneously available actuators at any point in the motion sequence and the joint resultant torque and force are distributed on the basis of some instantaneous measure of performance. For convenience the problem is typically broken into two parts, the determination of joint resultants and the muscle force distribution (Hardt, 1978).

From limb position histories, the hip, knee and ankle trajectories, and the corresponding angular velocities and angular accelerations are calculated. With these quantities known it is possible to solve the inverse dynamics problem to determine the joint moments required to generate the given kinematics (Bresler and Frankel, 1950). Given the joint moments at any instant, the problem then becomes that of finding the muscle forces that generate the moments. Since there are more unknowns in the problem than there are independent equations of motion, the problem at this point is indeterminate. To render the problem solvable, the unknown muscle forces are found using static optimization techniques based on a chosen muscle force or stress-based optimality criterion. This implies that the forces will be found at discrete points in time.

Calculating the joint resultant moments

There are two major methods for calculating the joint resultant moments given the kinematics of the problem. One method is by direct differentiation. In this method the angular positions are differentiated numerically to get the angular velocity and the angular accelerations. With these quantities known it is possible to calculate the moments required to generate the given kinematics. The equations of motion can be written in the form

$$ A(\dot{\theta})\ddot{\theta} = b(\theta, \dot{\theta}) + T \quad (31) $$

where $A(\theta)$ is the dynamic coupling matrix, $b(\theta, \dot{\theta})$ is the vector of dynamic moments and $T$ is the vector of applied moments.

Then given $\theta, \dot{\theta}, \ddot{\theta}$, $T$ can be calculated using the equation

$$ T = A(\dot{\theta})\ddot{\theta} - b(\theta, \dot{\theta}). \quad (32) $$

A second approach is to use an optimization method. In this technique the moments are assumed to be unknowns (decision variables) which are to be found in such a fashion that an objective function of the form

$$ J = \int_{t_0}^{t_f} H[\dot{\theta}(t) - \theta(t)]^2 \, dt \quad (33) $$

is minimized.

This technique was used by Chao and Rim (1973) to determine the hip, knee and ankle moments during a portion of the stance phase of gait. Although the problem appears to be a relatively simple one it leads to a well known problem of optimal control called singular optimal control. Our experience with this problem shows that it is difficult to solve using conventional algorithms for solving optimal control problems. Attempts to use other techniques such as those of Jacobson et al. (1970) did not alleviate the difficulties and often led to physiologically infeasible solutions (saturation of the controls).

Consequently, the differentiation technique was used to solve the inverse dynamics problem. To this effect, the Nearly Equal Ripple Derivative filter (NERD) designed by Kaiser and Reed (1977) was used and the smoothed derivatives were calculated by direct convolution.

Mathematical statement of the problem

The static optimization problem can be stated as follows. Minimize an objective function $J$ corresponding to a measure of muscle effort, and subject to the equality and inequality constraints corresponding to the joint moment equipollence relations, and the tensile nature of the muscle forces respectively

$$ \sum_{i=1}^{m} r_i \times f_i - T_j = 0 \quad j = 1, \ldots, 3 \quad (34a) $$

$$ f_i \geq 0 \quad i = 1, \ldots, m \quad (34b) $$

where $T_j$ is the resultant joint torque at the joint $j$, $r_i$ is the moment arm of muscle $i$, and $f_i$ is the force generated by muscle $i$.

One objective function proposed by Crowninshield and Brand (1981) on the basis of muscle fatigue considerations is the following

$$ J = \left[ \sum_{i=1}^{m} \left( \frac{f_i}{A_i} \right)^{1/3} \right]^{1/3} \quad (35) $$

where $A_i$ is the cross-sectional area of muscle $i$. Among a variety of candidates, this objective function has one of the more realistic physiological justifications and has been used in the present work.

The static optimization problem for the present model is then characterized by a total of eighteen independent variables (nine corresponding to the normalized muscle forces and nine slack variables introduced to transform inequality constraints to equality constraints). In implementing a solution for the static optimization procedure the muscle moment arms, $r$, in equation (34a) were calculated at each discrete time using the muscle lines of action based on
the prescribed origins and insertions and the instantaneous joint configurations.

**COMPUTER IMPLEMENTATION**

Computer programs were developed to implement the solutions for the dynamic and static optimization approaches. In solving the dynamic optimization algorithm a sampling interval of 0.0045 s was used. The whole swing phase was assumed to last 0.45 s. The state equations defined by equations (A1)-(A9) were integrated using the Hamming predictor-corrector method with the Runge-Kutta method for starting. This integration routine was chosen in order to minimize the number of function evaluations and also because of its stability characteristics. All quadratures were performed using the Simpson's rule. The optimization algorithm selected was the Fletcher-Reeves conjugate-gradient algorithm (Lasdon et al., 1967). The one-dimensional step size was found using the 'regula-falsi'-extrapolation-interpolation technique described elsewhere (Audu, 1985).

The static optimization problem was solved at 101 discrete time points which correspond to the discrete time points used in the dynamic optimization problem. The algorithm used was the static version of the Gradient-Restoration algorithm (Miele et al., 1969). By the definition of the problem the decision variables \( f_i \) are the muscle forces and hence have units of N. For enhanced numerical stability and computational efficiency, the problem was normalized by introducing dimensionless controls \( \bar{f}_i \) given by

\[
\bar{f}_i = f_i f
\]

where \( f \) is a scaling constant. An appropriate value for \( f \) was found to be 10 N.

**RESULTS**

(a) Results of the dynamic optimization

The specified hip, knee and ankle trajectories are depicted by the solid lines in Fig. 3 a–c. Shown on the same plots are the trajectories (dashed lines) predicted using the dynamic optimization approach. From these plots it can be appreciated that good tracking was obtained for most of the swing phase. The major discrepancies occur at the earlier parts where the generated trajectories tend to lag behind the specified trajectories.

Figures 4 a–c show the joint torques obtained by the dynamic optimization algorithm (solid lines) along with those obtained by solving the inverse dynamics problem using a particular set of filter characteristics subsequently described (dashed lines). The time histories of these curves are in general agreement except for the initial portions where the curves obtained by the dynamic optimization algorithm tend to lag those obtained by solving the inverse dynamics problem. This lag also appeared toward the end of the phase.

The control histories (stimulation rates) are shown in Fig. 5 a–c. The estimated envelopes of the EMGs given in Pedotti (1977) are also shown in Fig. 5 (dotted lines). Figures 6 a–i show the corresponding muscle force histories for the nine muscle groups studied. From these figures it can be appreciated that the initial flexor torque at the hip is realized by the activities of iliopsoas and rectus femoris muscle groups (Fig. 6a and c). These muscles remain active during the first 50–60% of the swing phase. The control histories for these muscles (Fig. 5a and c) show good temporal agreement with the EMG envelopes of Pedotti (1977). It should be mentioned that the EMG record for the iliopsoas is not available because of difficulties in measurement for this deep lying muscle group. The EMG envelope shown is estimated and therefore only an approximation. The hip extensor moment at the end of the swing phase is realized by the activities of gluteus maximus and the hamstring groups (Fig. 6i and b). This activity is necessary to bring the swinging limb to rest in preparation for heel strike.

A major part of the extensor moment at the knee is provided by the inertia of the swinging limb. The knee flexor torque at the end of the swing phase is accomplished by the activity of the hamstrings and the short head of biceps (Fig. 6b and c). A comparison of
Fig. 4. (a) Hip, (b) knee and (c) ankle joint torques obtained by dynamic optimization (solid lines). Dotted lines show the corresponding torques obtained by solving the inverse dynamics problem using the kinematic data of Fig. 8. Torque values are in Nm. The range for the abscissas is from toe-off to heel-strike.

Fig. 5. Control histories (normalized stimulation frequencies) in muscle groups 1–9 obtained by dynamic optimization. The possible range for the stimulation frequency for each muscle is \(0 \leq s \leq 1\). The range for each abscissa is from toe-off to heel-strike. EMG envelopes estimated by Pedotti (1977) are shown by dashed lines (no amplitude scale is implied).
Fig. 6. Predicted muscle forces in muscle groups 1-9 obtained by dynamic optimization. Muscle force values are all in N. The range for each abscissa is from toe-off to heel-strike.

The EMG envelopes and the predicted controls again show good temporal agreement. One interesting observation is that the vasti group (Fig. 5c) is predicted to have no activity at all during the swing phase. Contrary to this, the EMG records show that these muscles have some activity at the very end of the swing phase. The same is also true for the rectus femoris group (Fig. 5c). The activities of these muscle groups generally continue into the stance phase. The importance of this observation will be discussed subsequently.

Among the muscles that cross the ankle joint, only the tibialis anterior (Figs 5g and 6g) is active. The gastrocnemius (Figs 5d and 6d) and the soleus (Figs 5h and 6h) are silent throughout the swing phase. The EMG records of Pedotti (1977) for these muscle groups tend to confirm these predictions. The activity of tibialis anterior provides the necessary dorsiflexor torque at the ankle (Fig. 4c).

(b) Results of the static optimization

In order to evaluate the influence of NERD filter characteristics on the angular velocity and acceleration calculations, the acceleration data were integrated twice for comparison with trajectory data. In Fig. 7, parts a, b and c, the specified hip, knee and ankle trajectories are shown as solid lines. The dotted lines on the same figures show the corresponding trajectories obtained by integrating the accelerations twice. Parts d, e and f of the same figure show the corresponding angular velocities (solid lines), obtained by differentiating the specified trajectories using NERD (with parameters $N_p = 30$, $\beta = 0.08$ and $\delta = 0.01$; see
The dotted lines on the same figures show the angular velocities obtained by integrating the accelerations which are shown in parts g, h and i.

By varying the filter characteristics slightly, the results shown in Fig. 8 are obtained. In this case the filter characteristics were set at $N_p = 30$, $\beta = 0.05$ and $\delta = 0.01$ for the hip and knee trajectories, and $N_p = 30$, $\beta = 0.1$ and $\delta = 0.01$ for the ankle trajectory.

The corresponding joint torque and the muscle force histories predicted using the static optimization for the data of Fig. 7 are depicted in Figs 9 (solid lines) and 10 respectively. For the filtering results of Fig. 8, the corresponding results are shown in Figs 4 (dotted lines) and 11 respectively. The differences between the predicted force histories shown in Figs 10 and 11 are obvious. One sharp contrast is the predicted force history of the hamstring group (Figs 10b and 11b). In the first case, that muscle group was recruited only at the very beginning of the swing phase, to provide the small knee flexor moment shown in Fig. 9a. This contradicts the strong activity of this muscle group at the end of the swing phase as evidenced by its EMG record (Pedotti, 1977). This latter function was entirely left to the short head of biceps (Fig. 10c).

Figs 11a–i show the force histories obtained using the static optimization algorithm and filtering results in Fig. 8. From these figures it is observed that there is generally good agreement between these predicted force histories and those obtained by the dynamic optimization algorithm.
Fig. 8. Hip, knee and ankle kinematic data (a-c) Angular displacements, (d-f) velocities and (g-i) accelerations. Derivatives were obtained using NERD with $\beta = 0.05$ and $0.1$, $\delta = 0.01$, and $N_p = 30$. Dotted lines indicate curves obtained by integrating back from acceleration curves.

Fig. 9. (a) Hip, (b) knee and (c) ankle joint torques obtained by solving inverse dynamics problem with kinematic data of Fig. 7. Dotted lines show again the torques obtained by dynamic optimization shown in Fig. 4. Torques are in Nm. The range for the abscissas is from toe-off to heel-strike.
DISCUSSION

One of the prominent features in the results obtained by dynamic optimization is a characteristic lag between the input controls (Fig. 9) and the resultant force output (Fig. 10). This is mainly for the iliopsoas muscle group and the biceps femoris group. One of the consequences of this lag is that the joint torques produced by the dynamic optimization algorithm lag those produced by solving the inverse dynamics problem at the beginning of the swing phase. This difference also manifests itself in the predicted trajectories (Fig. 7). This early lag may be attributable to the fact that the problem starts at the beginning of the swing phase. Therefore preexisting muscle dynamics from the previous stance phase are not accurately accounted for in the early swing phase.

Such a characteristic lag between muscle input and muscle force output is one of the major differences between simple input-output muscle models necessarily assumed in the static optimization approach, and the dynamic muscle model which can be incorporated into the dynamic optimization approach. This lag is an inherent characteristic of the muscular subsystem which results from the differences in the response times of the electrical, chemical and mechanical aspects of contraction—making it impossible for the muscles to produce force instantaneously in response to stimuli.

Another interesting difference in the two results which is related to the muscle dynamics issue is that some of the muscle force predictions by static optimization exhibit sharp discontinuities (Fig. 11e). Such discontinuities are a direct consequence of the absence of memory in the static optimization. That is, the

**Fig. 10.** Predicted muscle forces in muscle groups 1–9 obtained by static optimization using kinematic data of Fig. 7. Muscle force values are in N. The range for the abscissas is from toe-off to heel-strike.
values of the forces obtained at any instant of time are independent of the values obtained at previous points in time. Such is not the case in dynamic optimization where the state and hence the control variables depend on the previous values obtained.

Another observation is that the muscle forces predicted by the dynamic optimization algorithm are generally larger than the corresponding forces predicted by the static optimization algorithm. This difference appears to be primarily because the passive joint structures (which include both elastic and damping elements) are included in the dynamic optimization algorithm but not in the static optimization algorithm. Available data for measured hip joint forces (Rydell, 1966; English and Kilvington, 1979) suggest that the static optimization methods have tended to predict somewhat higher joint forces than measured values (Crowninshield and Brand, 1981; Seireg and Arvikar, 1981). Thus it would appear that using the dynamic optimization approach, as well as modifying the static optimization approach to include passive joint resistances, would increase this discrepancy. However, it must be kept in mind that the peak joint forces occur during stance phase, which was not studied in the present work. Therefore, any conclusions about discrepancies must be tentative.

In considering differences between the results of the dynamic optimization and static optimization models, it is important to note the strong influence of the differentiation technique on the results of the static muscle force calculation. Although the comparisons between the better static solution (as judged from the re-integrated derivatives, Figs 7 and 8) and the dynamic solution were rather favorable, the comparisons based
on the differentiation results of Fig. 7 were much less so. This underscores the observations by others that a major source of error in the inverse dynamics problem is the numerical differentiation process (Hardt, 1978).

As a final discussion point it should be noted that the problem formulation and muscle/linkage models involve numerous idealizations. Among these are the restriction of the motion to a plane; the specified pelvic kinematics; the consideration of swing phase only in the gait cycle; the idealization of the joint structures and the lumping of the muscle groups. These choices were made on the basis of practical considerations and not any theoretical limitations of the approach. Two primary considerations were computational problem size and the specification of physical parameters such as muscle constants. In spite of these idealizations, it is felt that the problem is of a sufficient level of sophistication to examine the primary issue of interest, namely the significance of incorporating the dynamic muscle model into muscle force sharing predictions.

The muscle model itself is one of the important aspects of the dynamic optimization approach. It would be possible to further increase the complexity of this model in terms of both the number of control parameters and the complexity of the excitation and the contraction dynamics (Hatze, 1980). While the muscle model we have used is less sophisticated than is theoretically possible, the four element nonlinear representation incorporates several of the widely recognized features of muscle behavior (Audu and Davy, 1985). Certainly it is a substantial increase in sophistication from the static optimization or the simple input–output models, and it has allowed a first step toward understanding the significance of muscle dynamics in muscle force predictions.

REFERENCES


APPENDIX A. EQUATIONS FOR THE DYNAMIC OPTIMIZATION PROBLEM

The complete set of state equations for the dynamic optimization problem are given below. The vector of state variables...
includes the three link angles and the three link angular velocities, the normalized calcium concentrations and the normalized contractile element lengths for the nine muscles, and a parameter to incorporate the explicit appearance of the time \( t \). The dimension of the state vector is therefore 25. The dimension of the control vector of normalized stimulation frequencies is nine. In the equations, the index \( k \) corresponds to muscle number; \( k = 1, \ldots, 9 \).

\[
\begin{align*}
\dot{x}_1(t) &= f_1 = x_4 \\
\dot{x}_2(t) &= f_2 = x_5 \\
\dot{x}_3(t) &= f_3 = x_6 \\
\dot{x}_4(t) &= f_4 = x_7 \\
\dot{x}_5(t) &= f_5 = x_8 \\
\dot{x}_6(t) &= f_6 = x_9 \\
\dot{x}_7(t) &= f_7 = x_{10} \\
\dot{x}_8(t) &= f_8 = x_{11} \\
\dot{x}_9(t) &= f_9 = x_{12} \\
\end{align*}
\]

\[
\begin{align*}
\dot{x}_{10}(t) &= f_{10} + b_{51}x_{13} - x_{14} \\
\dot{x}_{11}(t) &= f_{11} + b_{52}x_{14} - x_{15} \\
\dot{x}_{12}(t) &= f_{12} + b_{53}x_{15} - x_{16} \\
\dot{x}_{13}(t) &= f_{13} = x_{17} = 1.
\end{align*}
\]

In these equations, for each \( k \)
\[
\begin{align*}
W_1 &= F_i - F_k^{SE} \\
W_2 &= (F_k^{SE} + b_{313}F_i) \\
W_3 &= F_i - F_k^{SE} \\
W_4 &= 1.33F_i - F_k^{SE} \\
F_1 &= b_{41}q(x_4 + 6)k(x_{12} + 15) \\
q(x_4 + 6) &= 1 - 0.995 \exp(-b_{31}x_{13}) \\
k(x_{12} + 15) &= \exp[-(x_{12} + 15 - 1)/b_{23}]K_2 \\
K_2 &= [1 - (b_{23}/b_{23}x_{14} + 15)]^{1/2} \\
F_k^{SE} &= f_{14}\exp(b_{14}K_2 - 1) \\
\delta_k &= L(k) - [(b_{32}x_{14} + 15)^2 - b_{32}^2](b_{32} - b_{6})
\end{align*}
\]

where \( L(k) \) is the length of the \( k \)th muscle. The quantity \( E \) in equation (19) takes the form
\[
E = Q_1 + Q_2 + Q_3 + Q_4
\]

where
\[
\begin{align*}
Q_1 &= a_{44}[a_{24}x_6 + H] \\
H &= F_i k(x_{12} + 15)q(x_4 + 6) \\
Q_2 &= b_{313}x_{13} + 15F_i b_{51} \\
Q_3 &= b_{52}x_{14} + 15F_i^{SE} \\
Q_4 &= c_5L^2(\delta).
\end{align*}
\]

The states \( x_1, x_2, x_3 \) represent the hip, knee, and ankle trajectories respectively; \( x_4, x_5 \) and \( x_6 \) are the corresponding angular velocities. \( x_{14} \) is the normalized calcium ion concentration. This quantity along with \( q \) defined by equation (A15) describe the excitation dynamics of the muscle. \( x_{14}(t) \) is the normalized length of the contractile element defined as the ratio of the length of the element and its isometric length. Equation (A8) is the muscle force-velocity relationship. \( s_\circ \) is the normalized frequency of stimulation and is the control variable in this paper. \( F_i \) defined by equation (A14) is the maximum isometric force of the muscle which is modulated by the active state of the muscle \( q \), defined by equation (A15), and the muscle force-length relationship \( k \), defined by equation (A16). \( F_k^{SE} \) defined by equation (A18) is the force in the series elastic element, and \( \delta \) defined by equation (A19) is the normalized extension of that element. The quantities \( Q_1 \) to \( Q_4 \) are the combined activation heat and maintenance heat rates, the shortening heat rate, the work rate and the rate of dissipation in the parallel structures respectively. The exact forms of the function \( f_{14}, f_k \) and \( f_k \) in the link dynamic equations (A4)-(A6), which represent the limb angular accelerations, are given in Audu (1985). The parameters \( b_{14}, b_{23} \) and \( a_{11}, a_{12} \) are given in Table 1. For complete details about the muscle modeling procedure and the techniques used to estimate the parameters see Audu and Davy (1985), Audu (1985).