The effect of muscle mechanics on human movement outcomes as revealed by computer simulation

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Abstract


Computer simulations were performed using a model of the human elbow joint as controlled by a single equivalent flexor muscle. The mechanical output of the muscle was determined from a user-specified neural drive according to the force–time, force–length, and force–velocity relations. The model allowed no storage of mechanical energy and no potentiation via the stretch reflex. The effects of different activation patterns as well as initial kinematic conditions on the maximum final velocity were examined. The results revealed a very nonlinear dependence of final velocity on both initial joint angle and angular velocity. Contrary to the principles of particle physics, it could be shown that an activation pattern that reached a maximum early in the movement achieved shorter movement times and, quite often, higher final velocities than an activation pattern that reached a maximum later in the movement. It was also found that by taking advantage of the nonlinear force–time, force–length and force–velocity relations, higher final velocities could be achieved if the muscle contracted from a previously stretch state with the absence of stored elastic energy. It was also found that with the same neural drive, the final velocities that could be achieved when the muscle was first required to absorb large amounts of energy in an eccentric contraction were similar to the final velocities that could be achieved when no negative work needed to be performed.

Introduction

Hochmuth and Marhold (1977) have stated that a movement with an acceleration pattern that rises to a peak very quickly and then decays less rapidly is a characteristic of a short movement time but not
a high final velocity. Conversely, a movement that has an acceleration pattern that rises less quickly to the same peak and then falls off rapidly is a characteristic of a movement with a high final velocity but not as short a movement time over a given distance. In general terms, this principle may best be illustrated using an example of a biphasic constant jerk. In this model, the acceleration starts at zero and linearly increases to a given maximum at time \( \tau \). The acceleration then linearly decreases to zero at time \( T \). If the distance remains constant, and the body starts at rest, the acceleration pattern that reaches the maximum value early will have a shorter movement time \( T \) but a lower final velocity than an acceleration pattern that reaches the same maximum later in the movement. The equations of this model are described in the Appendix and an example is shown in fig. 1.

These are provocative principles for human movement because many tasks can be classified as either requiring a maximum final velocity (i.e. shot put, high jump, etc.) or a minimum movement time (i.e. boxing punch). In situations where inertia is constant, the acceleration patterns are also those of the forces that need to be exerted and these patterns can be incorporated into the training protocol of practitioners who seek to enhance performance in these types of movement. While these principles of mechanics are undeniable, two clarifications must be made before they can be fully understandable as biomechanical principles.

The first clarification is that while both acceleration patterns have the same peak and therefore require the same magnitude of force, a muscle with the strength capable of generating the peak force is not necessarily capable of achieving both movement patterns. Fig. 2 shows the force functions derived from two acceleration patterns using the constant jerk model and assuming a constant inertia of 10 kg. The impression that many practitioners have of these principles is that if a muscle is capable of generating the force then it is simply a matter of choosing the appropriate activation pattern to achieve both movements. An activation pattern that reaches its maximum early in the movement would be assumed to cause a short movement time or the same muscle with an activation that reaches its maximum late in the movement would cause a high final velocity. It can be seen from the instantaneous power (product of force and velocity) that the movement that achieves the higher final velocity also required the capabil-
Fig. 1. Kinematics of a particle according to the acceleration functions given by the constant jerk model using two different times to reach maximum acceleration (solid line; \( \tau = 0.25 \) s) and \((+-+-; \tau = 0.75 \) s). Note that the acceleration pattern that reached the maximum first has a shorter movement time and a lower final velocity.
Fig. 2. Kinetics of a particle according to the two acceleration functions in fig. 1 given a mass of 10 kg. Note the greater instantaneous power required of the acceleration pattern that achieved the maximum value late in the movement.
ity of generating a much larger peak power. This notion is supported by data on the vertical jump which found that the peak instantaneous power was very highly correlated with take-off velocity but peak force was only weakly correlated (Dowling and Vamos 1991). Training that enhances the force generating capacity of muscle at slow speeds does not necessarily enhance force generating capacity at higher speeds (see Sale and MacDougall, 1981, for review).

The second clarification and the major focus of this paper is that depending on the mechanical state of the muscle, very different movement outcomes in terms of movement time and final velocity are achieved with the same activation pattern. The amount of force generated by a muscle is related in a nonlinear manner to the length of the muscle and the velocity of muscle shortening or lengthening. It is, therefore, unclear if a muscle activation pattern that reaches a peak early in the movement will always characterize a movement of short duration and if a later peak will always characterize a high final velocity.

Experiments performed on human subjects are often unable to control and quantify neural and mechanical variables well enough to clearly identify the reasons for the performance enhancement. In repeated voluntary contractions it is not possible to ensure that the neural activation of the many muscles involved remains the same in amplitude, timing and pattern. It is also not possible to control muscle contractile component lengths and velocities by standardizing joint angular kinematics due to the effects of series elasticity (Hof and Van Den Berg 1981).

The purpose of this study was to investigate the effect that different initial muscle lengths and velocities had on the movement outcomes via a computer simulation. In particular, two activation patterns were investigated in terms of movement time and final velocity. It was hypothesized that the principles of maximizing final velocity and minimizing movement time would not be as straightforward when complicated by the nonlinear mechanical response of muscle.

**Methodology**

In previous work, a computer model was developed and validated on human subjects for various elbow flexion and extension tasks
(Dowling and Norman 1988). The model combined the measurements of electromyography with the joint kinematics and fundamental elements of muscle mechanics (force–length, force–velocity, series elasticity) to predict active individual muscle forces crossing the elbow joint. The forces were then combined with an anatomical model and passive elements and finally validated with measured net joint moments of force during voluntary static and dynamic contractions.

A simplification of the earlier model was used for this study in which a single equivalent elbow flexor was used and the moment arm was fixed at 3 cm regardless of joint angle. There was no antagonist muscle and the movements were considered to act in the horizontal plane so that gravity was not a factor. The net moment of force or elbow torque was a consequence of the active muscle torque of the equivalent flexor only.

The muscle moment was determined from the activation of the muscle and the modulating effects of the force–velocity \((F-V)\) and force–length \((F-L)\) relations. Muscle activation was calculated using the Laplace transform transfer function, \(H(s)\), from a bang-bang neural drive. The user selected neural drive consisted of ten 60 ms segments each of which could have an amplitude of 0, 25, 50, or 100%. \(H(s)\) is the transfer function of a critically damped, second-order system with a time constant of 0.06 s. A pure delay of 0.01 s was used to simulate the electro-mechanical delay (Norman and Komi 1979). The gain \((K)\) was chosen such that a neural drive of 100% was calibrated to be equal to 100 Nm meaning that the maximum isometric elbow moment at the optimum muscle length was 100 Nm.

\[
H(s) = K\left(s + 1/T_i\right)^{-2},
\]

where:

- \(K\) – gain,
- \(T_i\) – time constant.

This transfer function reaches a peak in the impulse response after 60 ms. Previous investigators who have used critically damped transfer functions have used time constants of 0.05 s and 0.04 s for the biceps (Patla et al. 1982; Winter 1976, respectively). Other investigators have used overdamped second-order functions with time constants of 0.038 for the biceps (Crochetiere et al., 1967), 0.105 s for the soleus (Gottlieb and Agarwal 1971) and 0.1 s for the triceps (Crosby 1978).
Coggshall and Bekey (1970) found the time constants to vary considerably between subjects when the triceps were examined.

In comparison with experimental data, the bang-bang neural drive is analogous to full-wave rectified EMG and the transfer function is analogous to a low-pass filter with a cut-off frequency of 2.6 Hz. Fig. 3 shows examples of user-specified neural drives (rectangular lines) and the resulting activations or isometric muscle moments at optimum length.

The actual moment produced by a muscle is not simply a consequence of the activation. Muscle generates different forces at different lengths and at different velocities of shortening and lengthening (see fig. 4). The classic force–velocity \((F-V)\) relationship for shortening muscle is based on the work of Hill (1938) and the classic force–length \((F-L)\) relationship is based on the work of Gordon et al. (1966). In this study, a dimensionless moment-angular velocity relation
Fig. 4. Muscle moment modulating relationships based on the force–velocity (a) and force–length (b) relations of skeletal muscle.

has been adapted from Hill’s original equation based on the modelling results of Dowling and Norman (1988) to simulate both shortening and lengthening velocity effects on muscle moment at the elbow (see eq. (2)). The isometric condition is represented as zero velocity in fig. 4a and is associated with the modulation factor ($V_{FAC}$) of 1.0. This modulating factor decreases hyperbolically as the shortening velocity increases ($V > 0$) and increases hyperbolically as the lengthening velocity increases ($V < 0$). Shortening velocities greater than 12 rad/s are associated with a constant modulating factor of zero and lengthening velocities greater than 12 rad/s are associated with a constant modulating factor of 2.0. The constant ($C$) defined the concavity of
the function and was assumed to be 0.25 based on the excised mammalian muscle experiments reported by Close (1972).

\[
V_{\text{FAC}} = \frac{C \cdot V_0 \cdot (1 + C)}{C \cdot V_0 + V} - C, \quad \text{for } V \geq 0,
\]

\[
V_{\text{FAC}} = 2 - \left[ \frac{C \cdot V_0 \cdot (1 + C)}{C \cdot V_0 - V} - C \right], \quad \text{for } V < 0, \quad (2)
\]

where:
- \( V \) – angular velocity (rad/s),
- \( V_0 \) – maximum velocity (12 rad/s),
- \( C = 0.25 \).

A dimensionless moment–angle relation was adapted from the classic \( F-L \) relation of muscle and based on modelling results of Dowling and Norman (1988). The modulating factor of this relation (\( LF_{\text{AC}} \)) was calculated using eq. (3) and is shown graphically in fig. 4b. The shape of the function resembles that of a normal curve with the optimum length occurring at a joint angle of 110 degrees of extension (1.92 rad.).

\[
LF_{\text{AC}} = \exp \left\{ - \left[ \frac{\left( 1 - \frac{\theta}{110^\circ} \right)}{0.6} \right]^2 \right\}, \quad (3)
\]

where: \( \theta \) – joint angle in degrees.

The muscle moment was calculated at any instant in time to be the product of the activation (calibrated to Nm) and the two modulating factors based on the mechanical state of the muscle using eq. (4). There was no series elastic component or any other means of storing energy in a stretch–shortening cycle. There also was no means for enhancing the neural drive based on the stretch reflex or any other type of sensory information.

\[
T(t) = a(t) \cdot V_{\text{FAC}}(t) \cdot LF_{\text{AC}}(t), \quad (4)
\]
where:
\[ T(t) \quad - \quad \text{muscle moment at time } t \ (Nm), \]
\[ a(t) \quad - \quad \text{muscle activation at time } t \ (Nm), \]
\[ VFAC(t) \quad - \quad F-V \ modulating \ factor \ at \ time \ t \ (unitless), \]
\[ LFAC(t) \quad - \quad F-L \ modulating \ factor \ at \ time \ t \ (unitless). \]

The angular velocity achieved by a limb at a certain joint angle was dependent on the moment applied at the joint and the moment of inertia of the limb. In this simulation the only moment applied at the joint was the muscle moment. The simulations were performed on a digital computer at a step size of 2 ms. The elbow angular positions and velocities were determined by dividing the muscle moment by the moment of inertia and integrating via the trapezoidal rule using various initial conditions.

In the first experiment the effects of initial joint positions and angular velocities were examined in terms of achieving a maximum final velocity. The moment of inertia was selected to be 0.29 kg.m\(^2\) which corresponds approximately to a 70 kg subject holding a 2 kg mass in the hand. The neural drive also remained constant as a maximal burst lasting 120 ms and is shown as the rectangle in the top graphs of fig. 6. The simulation was run repeatedly with different initial angular positions ranging from 180 degrees to 50 degrees in 2 degree steps and with different angular velocities ranging from \(-12\) rad/s to \(12\) rad/s in 0.5 rad/s steps. The negative velocities refer to extension of the elbow and, therefore, to muscle lengthening.

In the second experiment, the effects of activation pattern and initial position were investigated at two different initial velocities. The moment of inertia was selected to be 0.3 kg.m\(^2\) and the two initial velocities were \(0\) rad/s and \(-3\) rad/s. The two activation patterns were chosen such that one achieved maximum activation early and the other late in the movement. In both cases the neural drive lasted 180 ms (three steps of 60 ms each) but the first pattern had a sequence of 100–50–25\% and the other had a sequence of 25–50–100\%. The bang-bang neural drives and activations of these two strategies is shown in fig. 5. It can be seen that in both cases the total neural drive is the same and the area under the resulting activation-time curves are, therefore, also the same. The first pattern achieves a peak of 48\% at a time of 0.15 s while the second achieves about 54\% activation in more than 0.2 s.
Fig. 5. Activation patterns (smooth lines) from the bang-bang neural drive sequences of 100–50–25% (top) and 25–50–100% (bottom).

To further demonstrate the stretch–shortening cycle enhancement in the absence of mechanical energy storage and neural potentiation, two additional experiments were performed. In the third experiment, a moment of inertia of 0.07 kg.m² (similar to a 70 kg subject in an unloaded situation) was chosen. A 120 ms maximal neural drive was used and an initial angle of 90 degrees was chosen. Two simulations were performed; the first with an initial velocity of zero rad/s and the second with an initial velocity of −6 rad/s.

Previous investigators examining depth or drop jumping have found that the height that jumpers can achieve following drops from greater and greater heights increases up to a point after which the performance decreases (Komi and Bosco 1978). The increased performance has been attributed to the utilization of energy that was stored in the stretch. A fourth experiment was performed to examine this effect in
a situation where the muscle could not store energy. A moment of inertia of 0.8 kg.m² was chosen with the same neural drive as before and a constant initial position of 70 degrees. A series of simulations were then performed changing the initial velocity from 2 rad/s to -12 rad/s and the final velocities (at 50 degrees) were recorded as in the previous experiment.

Results

The first column of graphs in fig. 6 shows the resulting moment and kinematics of the limb ignoring the effects of the $F-V$ and $F-L$ relations. It can be seen that in response to the neural drive, the muscle moment rises rapidly following the short electromechanical delay (10 ms) to a maximum of 65 Nm in about 160 ms. If the neural drive was maintained, the moment would eventually reach a plateau of 100 Nm. The high moment causes the limb to accelerate rapidly reaching an angular velocity far greater than 12 rad/s (see broken lines). The limb started in the fully extended position (180 degrees) and was passively stopped at full flexion (40 degrees) in about 250 ms. The stop caused the velocity to instantly return to zero even though the muscle was still generating a moment. This first column is an unrealistic situation because the actual moment developed by the muscle is subject to the modulating effects of length and velocity discussed earlier.

The second column of graphs in fig. 6 shows the resulting moment and kinematics of the limb if the activation and $F-L$ relations are considered. Since the limb started in the extended position, the muscle length was longer than the optimal and the resulting moment is less than in the first column. As the angular position approached 110 degrees (optimum muscle length) at about 300 ms, the moment converged toward that of the first column but then dropped off again as the muscle became shorter than the optimum length. The lower moments caused less acceleration but the movement reached full flexion in about 360 ms and again achieved a velocity beyond the 12 rad/s plotting scale before being stopped.

The third column of graphs in fig. 6 shows the resulting moment and kinematics of the limb if the activation and $F-V$ relations are considered. As the shortening velocity of the limb increased the
Fig. 6. Kinetic and kinematic results of the same neural drive considering the force–time, force–length, and force–velocity relations separately (first, second, and third columns, respectively) and combined (fourth column).
Fig. 7. The final velocity surface (angular velocity when the limb reaches full flexion) given different initial positions and velocities of a limb with a neural drive of 100% for 120 ms and a moment of inertia 0.29 kg.m².

The muscle moment was reduced more and more below those in the first column. This movement did not reach full flexion for about 450 ms and had reached an angular velocity of over 7 rad/s before being stopped.

The fourth column is the most realistic in terms of the actual moment and kinematics produced by such an activation starting at rest in the fully extended position. This column combines the activation with both the $F-L$ and $F-V$ relations. It can be seen that since the muscle is not at optimum length when the velocity is zero and that since the velocity is not at zero when the muscle is at optimum length, the muscle moment is always less than in the first column. The lower moment causes the limb to reach a maximum velocity of less than 6 rad/s and to require more than 600 ms to reach full flexion.

The final velocity achieved at full flexion (50 degrees) is dependent on the initial position and initial velocity in a very nonlinear fashion even when the activation and moment of inertia remain constant. Fig. 7 shows the results of repeated simulations using the same moment of inertia and neural drive as in fig. 6. The surface shows the nonlinear
dependence of this final velocity on each combination of starting position (ranging from 180 to 50 degrees) and velocity (ranging from −12 to 12 rad/s). It can be seen that the best joint position to start from differs depending on the angular velocity of the limb. It is also interesting to note the trough that starts off quite shallow at the extended initial position at zero velocity and then becomes progressively deeper toward more intermediate initial positions as the initial velocity becomes larger in the negative direction.

It should be emphasized that the surface plot shown in fig. 7 is for a certain activation and a certain moment of inertia. Another activation pattern would result in an entirely different shaped surface. Increases or decreases in the moment of inertia will not only change the magnitudes of the final velocities but will also change the shape of the surface.

The results of the second experiment show that when starting from rest, the activation pattern that reached its maximum late in the movement achieved the higher final velocity only when the initial angle was larger than 125 degrees. These differences were, however, very small and the two lines are almost indistinguishable in fig. 8 (top). If the muscle was initially lengthening, the activation pattern that reached its maximum later in the movement had a higher final velocity at almost all starting angles except in the region of the trough (see fig. 8 – top).

Regardless of the initial joint angle, the activation pattern that reached its maximum early in the movement required less time to reach full flexion than the pattern that reached maximum activation later (see fig. 8 – bottom). This effect became even more pronounced at the shorter joint angles when the muscle was initially lengthening.

The results of the two additional experiments indicated that the stretch–shortening cycle could be demonstrated in a situation where there was no utilization of previously stored elastic energy or neural potentiation via the stretch reflex. Fig. 9 shows the two trials of the third experiment. Both columns started at the 90 degree joint position but in the first column, the limb started from rest and in the second column the limb started with a velocity of −6 rad/s. Due to the low inertia of the limb and the intermediate starting position, the movement was completed in a very short period of time. It can be seen in both columns that the movement was over while the muscle was still generating torque. The second column which had the muscle contract-
Fig. 8. The final velocities (top) and movement times (bottom) of a limb with a moment of inertia 0.8 kg.m^2 given two patterns of neural drive and two initial velocities. The neural drives of 100–50–25% are represented by the squares and diamonds and the neural drives of 25–50–100% are represented by the triangles and plus signs. The initial velocities of zero are represented by the squares and plus signs and the initial velocities of −3 rad/s are represented by the diamonds and triangles.

...ing eccentrically at the beginning of the movement allowed the muscle torque to rise much more quickly than in the first column. This counter-movement which lasted about 100 ms also allowed the muscle to reach lengths that were more optimal (joint angle of 110 degrees) than those lengths reached in the first column. The counter-movement also caused the movement to last longer preventing less wastage of muscle torque after the movement had been stopped and allowed the limb to achieve a position of near optimal length at a more optimal velocity than in the first column. Each of these factors enabled the muscle to perform the necessary negative work to stop the...
extension and achieve a higher final flexion velocity with the same neural drive. The final velocity of the second column was 10.2 rad/s while the first column was only 9.4 rad/s.

The results of the fourth experiment are shown in fig. 10. It can be seen that as the initial velocity became more negative the final velocity increased up to a maximum (−8 rad/s) and then decreased. This in accordance with findings of Komi and Bosco (1978) who found that the take-off velocity of vertical jumpers increased with the height from which their subjects dropped until the height exceeded 62 cm. The reason for the decline in final velocity was due to the joint becoming quite extended by the time the muscle could perform enough negative
work to bring the limb to rest. Much of the activation was then wasted at joint angles that did not allow large forces to be generated and, therefore, a lower final velocity was achieved.

Discussion

The movement outcomes of a human limb given a certain neural drive and moment of inertia is very much dependent on the initial dynamic situation of the limb in terms of position and velocity. Even though the simulation used in this study was a gross simplification of the actual human elbow, the complicated interactions of the nonlinear force–time, force–velocity, and force–length relations make the selection of the optimal movement strategy very difficult. Knowing that a muscle can generate its maximum isometric moment at a joint angle of 110 degrees does not mean that 110 degrees is the best angle from which to start the movement. Due to the force–time relation of muscle, by the time the muscle is at maximum activation, it will no longer be at the optimum angle.

In order to choose the starting position that achieves the highest final velocity given a certain pattern of activation, one must also know the initial velocity of the limb. When starting from rest, the optimal starting angle may be 142 degrees (i.e. fig. 7) but as the initial velocity
increases in the negative direction the optimal starting angle decreased to 78 degrees. This has important implications for the instruction of movement techniques. The reason for a poor performance may not be due so much to an incorrect starting position as to an incorrect starting velocity for that position. A coach may instruct a novice athlete to change the elbow position to an angle that should be more optimal based on comparisons with elite athletes only to find the performance to be decreased because the initial velocity of the novice elbow was not similar to that of the elite.

It is possible for a joint to have a positive angular velocity prior to activation of the flexor muscle due to the transfer of energy from an adjoining segment that is losing energy (Pierrynowski et al. 1980). This phenomenon is illustrated in fig. 7 where the initial velocity is greater than zero. As the initial velocity increased, the influence of the initial angle decreased as evidenced by the progressive flattening of the surface towards the high initial velocities. In the extreme case when the limb had an initial velocity of 12 rad/s, the muscle was shortening at a velocity that allowed zero force to be generated even though the muscle was active. Regardless of the initial angle, therefore, the final velocity was always 12 rad/s. When the limb had an initial velocity that was negative, the flexor muscle was lengthening in an eccentric contraction before bringing the limb to rest and then proceeded with a concentric contraction. This stretch–shortening cycle was examined in more detail in the third part of this study.

The impulse response of a critically damped second order system is characterized by a rapid increase followed by a more slow exponential decrease. This pattern, as described in the introduction, is a characteristic of the acceleration pattern associated with a short movement time rather than a high final velocity. It may have been expected, therefore, that the activation pattern that reached a maximum early in the movement would be associated with shorter movement times and that the activation pattern that achieved a maximum late in the movement would be associated with higher final velocities. Such was not always the case. The movement times were always shorter with the activation pattern that reached a maximum early in the movement but the expected improved final velocity of the pattern that achieved maximum activation later in the movement was not seen. There was very little difference between the two patterns when the limb started from rest but the activation pattern that reached a maximum early in
the movement had higher final velocities when the movement started at \(-3\) rad/s.

Contrary to what might have been expected from the mechanics principles discussed in the introduction, when it comes to activating muscles to achieve certain movement outcomes, there may not be a trade-off between achieving high final velocities and low movement times. For most of the initial joint angles, the activation pattern that achieved a maximum early in the movement also achieved a final velocity that was as high or higher than the activation that reached a later maximum. In many ballistic human movements (i.e. high jump), the elite performance is characterized by both higher final velocities and shorter movement times than the novice (Klissouras and Kar-povich 1967).

It should be emphasized that only two of an infinite number of possible activation patterns were examined in this study and that different moments of inertia and activations would produce results that have effects that are either greater than or less than those shown here. If the human movement practitioner wishes to include techniques into a training regime that are designed to alter activation patterns to improve performance, it is not enough to treat the limb as a particle being acted upon by an ideal force generator. The force generator (muscle) is greatly affected by the force–time, force–length, and force–velocity relations and the movement outcomes are, therefore, also affected.

The intent of the third and fourth experiments of this study was not to belittle the very excellent work on the role of elastic energy storage or of stretch reflex potentiation on the enhancement of the stretch–shortening cycle. The findings of this study do suggest that the force–time, force–length, and force–velocity relations also play a role in the performance enhancement of the stretch–shortening cycle. The exact magnitude of that role awaits more accurate and valid musculoskeletal models of the human system.

Conclusions

While the model of the human elbow joint that was used in this study was an over-simplification of the real system, it was used to demonstrate the highly complex and nonlinear interactions of the
different mechanical variables and their effects on human movement outcomes. The actual values may, therefore, not be strictly valid but the trends and characteristics that they revealed are. The optimum position from which to begin a movement is very much dependent on the initial velocity. An activation pattern that reaches a maximum early in the movement is not only associated with a shorter movement time but, in many cases, a higher final velocity as well. As stronger and stronger validations of musculoskeletal models are achieved, the better computer simulations will be at identifying optimal movement patterns and strategies for training.

Appendix

A biphasic, constant jerk model as shown in fig. 1 of the text is defined mathematically below in eqs. (A.1a) and (A.1b). Jerk is defined as the third derivative of displacement. Eq. (A.1a) represents the positive jerk phase and eq. (A.1b) represents the negative jerk phase.

\[ j(t) = j_c, \quad 0 \leq t < \tau \]  
\[ j(t) = -\frac{\tau}{T-\tau} \cdot j_c, \quad \tau < t \leq T \]

where:
- \( t \) – elapsed time of the movement,
- \( j(t) \) – jerk as a function of time,
- \( j_c \) – constant jerk,
- \( \tau \) – duration of positive jerk phase,
- \( T \) – total time of the movements.

Choosing initial conditions of acceleration, velocity and displacement to be zero and integrating with respect to time yields the following equations of motion for the positive jerk phase. The final acceleration at time \( T \) has also been chosen to be zero.

\[ a(t) = j_c \cdot t, \]  
\[ v(t) = \frac{j_c}{2} \cdot t^2, \]
where:

\( a(t) \) – acceleration with respect to time,
\( v(t) \) – velocity with respect to time.

Choosing a final acceleration at time \( T \) to be zero and integrating
the negative jerk phase with respect to time yields the following
equations:

\[
a(t) = j_c \cdot \frac{T - t}{T - \tau}. \quad \text{(A.3a)}
\]

\[
v(t) = \frac{j_c}{2} \cdot \frac{T - (T - t)^2}{T - \tau}. \quad \text{(A.3b)}
\]

The maximum acceleration occurs at \( t = \tau \) and is given by:

\[
a_{\text{max}} = j_c \cdot \tau \quad \text{(A.4)}
\]

where: \( a_{\text{max}} \) – maximum acceleration.

The constant jerk \( (j_c) \) is dependent on the maximum acceleration
and the time required to reach maximum acceleration \( (\tau) \). By rear-
ranging eq. (A.4), the constant jerk can be calculated when the
maximum acceleration and the duration of the positive jerk phase are
known using:

\[
j_c = \frac{a_{\text{max}}}{\tau}. \quad \text{(A.5)}
\]

The final distance is given by:

\[
d_f = \frac{j_c}{6} \cdot \tau \cdot T \cdot (2T - \tau), \quad \text{(A.6)}
\]

where: \( d_f \) – final distance at \( t = T \).

By choosing a fixed maximum acceleration and constraining the
movement to occur over a given distance, it can be seen that by
Fig. A.1. Movement times (top) and final velocities (middle) of the constant jerk model with a maximum acceleration of 5 m/s² and a constant displacement of 1 m. Note the nearly linear increase in movement time and final velocity with increased time to maximum acceleration ($\tau$). The maximum instantaneous power requirement (bottom) dramatically increases as the time to reach maximum acceleration increases.
manipulating $\tau$, different movement times ($T$) and final velocities ($v_f$) will be calculated using eqs. (A.7) and (A.8), respectively.

$$T = \frac{\sqrt{\tau^2 + 48 \cdot d_f}}{j_c \cdot \tau}, \quad (A.7)$$

$$v_f = \frac{j_c \cdot \tau}{2} \cdot T, \quad (A.8)$$

where: $v_f$ – final velocity.

An example is given where the movement distance and maximum accelerations are fixed at 1 m and 5 m.s$^{-2}$, respectively. The results are shown in fig. A.1 for $\tau$ values ranging from zero up to the maximum which is given by:

$$\tau_{\text{max}} = \frac{\sqrt{6 \cdot d_f}}{a_{\text{max}}}, \quad (A.9)$$

where: $\tau_{\text{max}}$ – maximum allowable duration of the positive jerk phase.

The instantaneous power was calculated by multiplying acceleration by mass to get force and then multiplying force by velocity to get power. This calculation is given by:

$$p(t) = m \cdot a(t) \cdot v(t), \quad (A.10)$$

where:

$p(t)$ – instantaneous power at time $t$,

$m$ – mass.

To calculate the maximum power required by the movement, eq. (A.10) was differentiated with respect to time and equated to zero to get the time of maximum power. This time value was then substituted back into eq. (A.10) to yield the maximum instantaneous power.

It can be seen from fig. A.1 that as the time to reach maximum acceleration ($\tau$) increases, the time to complete the movement (fig. A.1 – top) and the final velocity (fig. A.1 – middle) also increase. Movements requiring short movement times should achieve the maxi-
mum acceleration as soon as possible and those requiring a high final velocity should reach maximum acceleration as late as possible. The power requirements are dramatically increased (fig. A.1 – bottom) as the time to reach maximum acceleration increases.

References