Cocontraction of Pairs of Antagonistic Muscles: Analytical Solution for Planar Static Nonlinear Optimization Approaches

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ABSTRACT

It has been stated in the literature that static, nonlinear optimization approaches cannot predict coactivation of pairs of antagonistic muscles; however, numerical solutions of such approaches have predicted coactivation of pairs of one-joint and multijoint antagonists. Analytical support for either finding is not available in the literature for systems containing more than one degree of freedom. The purpose of this study was to investigate analytically the possibility of cocontraction of pairs of antagonistic muscles using a static nonlinear optimization approach for a multidegree-of-freedom, two-dimensional system. Analytical solutions were found using the Karush-Kuhn-Tucker conditions, which were necessary and sufficient for optimality in this problem. The results show that cocontraction of pairs of one-joint antagonistic muscles is not possible, whereas cocontraction of pairs of multijoint antagonists is. These findings suggest that cocontraction of pairs of antagonistic muscles may be an "efficient" way to accomplish many movement tasks.

INTRODUCTION

Static, nonlinear optimization approaches have been used frequently [4, 7–9, 11, 16] to solve the so-called distribution problem in biomechanics [5]. Nonlinear optimization approaches have various advantages over linear optimization approaches. For example, they will predict coactivation of agonistic muscles without having to impose upper-limit constraints on muscular forces [3]. Furthermore, it has been shown analytically that static, nonlinear optimization approaches can also
predict antagonistic muscular activity [10], but the issue of cocontraction of pairs of antagonistic muscles is still controversial. For example, Hughes and Chaffin [12], using the approach of Crowninshield and Brand [4], reported that cocontraction of pairs of antagonistic muscles spanning a single joint is not possible in a two-dimensional system, whereas Davy and Audu [6] exhibit such cocontraction between soleus and tibialis anterior muscles.

The purpose of this study was to investigate analytically the possibility of cocontraction of pairs of antagonistic muscles using the static nonlinear optimization approach introduced by Crowninshield and Brand [4]. This approach requires that the sum of all muscular stresses to a power larger than 1.0 be minimized, that all resultant external joint moments be equilibrated by muscular moments, and that muscular forces be only tensile (i.e., unidirectional). The optimization approach of Crowninshield and Brand [4] was chosen because most of the controversy about cocontraction of antagonistic muscles has revolved around this particular algorithm [6, 10, 12, 15, 18]. All calculations were made using a two-dimensional, two-degree-of-freedom musculoskeletal model that contained four single-joint and two two-joint muscles (Figure 1). A two-dimensional system was chosen because antagonistic muscular ac-

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**Fig. 1.** Musculoskeletal model used for all calculations.
tivity has been defined uniquely for such systems [1, 10] and cocontrac-
tion of pairs of antagonistic muscles may be defined as a logical
extension of the definition of antagonism. There is no universally
accepted definition of antagonism for three-dimensional systems, but
we shall comment on how our results bear on such systems at the end of
this article. A two-degree-of-freedom system was chosen because the
problem of cocontraction has been investigated before for a one-de-
gree-of-freedom system [12] and a two-degree-of-freedom model con-
taining two-joint muscles reflects biological reality more adequately
than a one-degree-of-freedom model. Furthermore, it was expected that
a two-degree-of-freedom system would possess different characteristics
from those of a one-degree-of-freedom system.

ONE-JOINT MUSCLES FOR GENERAL SYSTEMS

Consider a general musculoskeletal system modeled as a frictionless
pin-jointed framework of bones connected by muscles with forces \( f_i \)
(Figure 1). Equating force moments to zero at each joint leads to a
system of linear equations of the form

\[
A f = b
\]

where \( A \) is the matrix of moment arms and \( b \) is the (column) vector of
external resultant joint moments. The object is to minimize a continu-
ously differentiable "cost function" \( c \) of \( f \) subject to (1) and nonnegativ-
ity of the \( f_i \). Then the Karush-Kuhn-Tucker optimality conditions apply.
(See [14], Section 5.1. Specifically, we write Equation (1) in inequality
form \( A f < b \) and \( -A f < b \) so the so-called constraint qualification
holds by Lemma 5.2(a). Then Theorem 5.1 gives the Karush-Kuhn-
Tucker conditions, which take the form

\[
\nabla c = A^T \lambda + \mu, \tag{2}
\]

where \( T \) denotes transpose and \( \lambda, \mu \) are Lagrange multipliers satisfying
\( \mu_i \geq 0 \) and \( \mu_i f_i = 0 \) for each \( i \).

If \( f_m \) and \( f_n \) form a (one-joint) antagonist pair at the \( j \)th joint, then
\( f_m \) and \( f_n \) appear in only the \( j \)th row of (1). Thus the \( m \)th and \( n \)th rows
of (2) take the form

\[
\frac{\partial c}{\partial f_m} = \alpha \lambda_j + \mu_m \quad \text{and} \quad \frac{\partial c}{\partial f_n} = \beta \lambda_j + \mu_n,
\]

respectively, where \( \alpha \beta < 0 \). If both \( f_m \) and \( f_n \) are positive, then
\( \mu_m = \mu_n = 0 \), and we obtain \( \beta (\partial c / \partial f_m) = \alpha (\partial c / \partial f_n) \). Thus, if \( \partial c / \partial f_m \)
and $\partial c / \partial f_n$ have the same sign, we obtain a contradiction. This proves the following theorem.

**THEOREM 1**

*If, for a one-joint antagonist pair of muscles with corresponding forces $f_m$ and $f_n$, the cost function $c$ to be minimized satisfies

\[
\frac{\partial c}{\partial f_m} \frac{\partial c}{\partial f_n} > 0
\]

whenever $f_m$ and $f_n$ are positive, then cocontraction of this antagonist pair is impossible.*

We remark that this result extends easily to allow upper-bound constraints on the $f_i$. In this form, our result extends the one stated by Hughes and Chaffin [12] to general configurations with any number of joints. We also remark that Hughes and Chaffin [12] omit their proof, but the reference they cite (Luenberger [13]) requires linear independence of the active constraints, and this does not hold in general. For example, if a joint moment is satisfied by two muscles and that joint moment can be satisfied only for values equal to the upper-bound constraints for the two muscles, then the active constraints (i.e., the moment constraint and the two inequality constraints limiting the muscle forces to their upper bounds) are not linearly independent.

**A TWO-JOINT SYSTEM**

The theoretical musculoskeletal system used for the remainder of this investigation consisted of a two-segment, two-joint, six-muscle two-dimensional model (Figure 1), and muscular forces were calculated using the static, nonlinear optimization approach proposed by Crowninshield and Brand [4]. Pairs of antagonistic muscles were defined as muscles crossing the same joint(s) and producing opposite moments about each joint they cross. In this model, antagonistic pairs are made up of muscles 1 and 2, muscles 3 and 4, and muscles 5 and 6. Numerical solutions were obtained using standard search procedures. Analytical solutions were calculated using the Karush-Kuhn-Tucker conditions, which are necessary, as in (2), and also sufficient because the objective function of Crowninshield and Brand [4] is strictly convex in the constraint set for $\alpha > 1.0$, $f_j > 0$, and $\text{PCSA}_j > 0$, and the constraint equations are componentwise linear [2, Theorem 4.38]. (Actually we set $c = \infty$ outside the constraint set and use one-sided derivatives in the cited reference. For this technique see, for example, [17, Section 28]). Here $\alpha$ is the power of the objective function, and $f_j$ and $\text{PCSA}_j$ are
the muscular forces and corresponding physiological cross-sectional areas, respectively. All solutions were calculated by imposing an external force on the distal segment of the model that produced moments about the proximal (M1) and distal joints (M2) of the system.

Muscular forces were determined for static equilibrium conditions and assuming frictionless hinge joints. Moment arms $r_j$ and physiological cross-sectional areas PCSA$_j$ were initially taken to be 1 arbitrary unit. Deviations of these parameters from unity are discussed subsequently.

AN OPTIMIZATION PROBLEM

The cost function to be minimized is given by

$$c(f) = \sum_{i=1}^{6} \left( \frac{f_i}{PCSA_i} \right)^\alpha$$

subject to $f_i > 0, i = 1, \ldots, 6$.

Let us assume that the muscular moments to be satisfied are $M1 = -1$ (arbitrary unit) at the proximal joint and $M2 = 1$ (arbitrary unit) at the distal joint. A positive moment is defined to be counterclockwise. Furthermore, it is assumed that all moment arm values are 1 (arbitrary unit), with the exception of the moment arm of muscle 6 about the proximal joint, which will be designated as $\beta$. Therefore, it follows that (1) takes the form

$$-f_1 + f_2 - f_5 + \beta f_6 = -1 \quad \text{and} \quad -f_3 + f_4 - f_5 + f_6 = 1. \quad (3)$$

OPTIMALITY CONDITIONS

For this system, with $\alpha > 1.0$ and $0 < \beta < 1.0$, it will be shown that $f_1, f_3, f_5, f_6 > 0$ and $f_2 = f_4 = 0$. Therefore, there is cocontraction for the pair of two-joint antagonistic muscles, $f_5$ and $f_6$. It is easily seen that the conditions of Theorem 1 are satisfied, so the one-joint pair corresponding to $f_1, f_2$ cannot cocontract and neither can the pair corresponding to $f_3, f_4$.

For this system, the optimality conditions [16] are

$$\alpha f_1^\alpha - 1 = -\lambda_1 + \mu_1, \quad (4)$$
$$\alpha f_2^\alpha - 1 = \lambda_1 + \mu_2, \quad (5)$$
$$\alpha f_3^\alpha - 1 = \lambda_2 + \mu_3, \quad (6)$$
$$\alpha f_4^\alpha - 1 = \lambda_2 + \mu_4, \quad (7)$$
$$\alpha f_5^\alpha - 1 = -\lambda_1 - \lambda_2 + \mu_5. \quad (8)$$
and

\[ \alpha f_6^{a-1} = \beta \lambda_1 + \lambda_2 + \mu_6, \]  
\[ \text{where all } \mu_i > 0 \quad \text{and} \quad \mu_i f_i = 0. \]  

\[ (9) \]

\[ (10) \]

**DEMONSTRATION OF COCONTRACTION**

First, suppose that \( f_1 = 0 \), so \( \lambda_1 = \mu_1 \geq 0 \) from Equation (4). Equations (10) and (3) then give \( f_2 - f_3 + \beta f_6 = -1 \), giving \( f_3 = 1 + f_2 + \beta f_6 > 0 \). Therefore, \( \mu_3 = 0 \) by Equation (10), so \( \lambda_1 + \lambda_2 < 0 \) by Equation (8), and we can conclude that

\[ \lambda_2 < 0. \]  

\[ (11) \]

Clearly, Equation (7) does not allow \( f_4 > 0 \), since then \( \mu_4 = 0 \) by Equation (10), so \( f_4 = 0 \). Equation (3) then gives \( f_6 = 1 + f_3 + f_5 > 0 \); so by (9), \( f_4 = 0 \) implies

\[ \alpha f_6^{a-1} = \beta \lambda_1 + \lambda_2 > 0. \]  

\[ (12) \]

Since \( 0 < \beta < 1 \), we know that \( \beta (\lambda_1 + \lambda_2) < 0 < \beta \lambda_1 + \lambda_2 \), so \( \lambda_2 > 0 \), which is in contradiction with (11). From Theorem 1 we conclude that

\[ f_1 > 0 = f_2. \]  

\[ (13) \]

Second, suppose that \( f_4 = 0 \), so \( \lambda_2 = -\mu_4 \leq 0 \) by Equation (7), and thus (12) gives \( \lambda_1 > 0 \). Since \( f_1 > 0 \), \( \mu_1 = 0 \) by Equations (4), (10), and (13), there is a contradiction with Equation (4). Using our theorem again, we conclude that \( f_4 > 0 = f_3 \), and from Equations (4), (7), (10), and (13) we deduce

\[ 0 < \alpha f_1^{a-1} = -\lambda_1, \quad 0 < \alpha f_4^{a-1} = \lambda_2. \]  

\[ (14) \]

Third, suppose that \( f_5 = 0 \), so by Equations (8) and (14) we obtain

\[ \alpha (f_4^{a-1} - f_5^{a-1}) = \lambda_1 + \lambda_2 = \mu_5 > 0, \quad \text{so} \quad f_4 > f_1. \]  

However, Equation (3) gives \( f_1 = \beta f_6 + 1 > 1 \) and \( f_4 = 1 - f_6 \leq 1 \), so \( f_1 = f_4 = 1 \), giving \( f_6 = 0 \) and \( \beta \lambda_1 + \lambda_2 = -\mu_6 \leq 0 \) (\( \leq \lambda_1 + \lambda_2 \)). This gives \( \lambda_1 > 0 \), contradicting (14), and so \( f_5 > 0 \).

Finally, suppose that \( f_6 = 0 \), so \( \beta \lambda_1 + \lambda_2 = -\mu_6 \leq 0 \). Thus (14) gives \( f_4^{a-1} - \beta f_1^{a-1} \leq 0 \), whence \( f_4 < \beta^{1/(a-1)} f_1 < f_1 \). However, Equation (3) yields \( f_1 = 1 - f_5 < 1 \) and \( f_4 = 1 + f_5 > 1 \), so we have a contradiction and accept that \( f_6 > 0 \).

In summary, for the system specified above, the optimal solution satisfies \( f_1, f_4, f_5, f_6 > 0 \) and \( f_2 = f_3 = 0 \).
An Example. Let $\beta = 0.5$ and $\alpha = 2$. From Equations (3), (4), (7), (8), and (9), we then have four linear equations:

\begin{align*}
-f_1 - f_5 + \beta f_6 &= -1, \\
f_4 - f_5 + f_6 &= 1, \\
f_1 - f_4 &= f_5, \\
-\beta f_1 + f_4 &= f_6.
\end{align*}

(15)  
(16)  
(17)  
(18)

Solving these equations, we obtain the unique solution

\begin{align*}
f_1 &= 1, \\
f_2 &= 0, \\
f_3 &= 0, \\
f_4 &= 5/6, \\
f_5 &= 1/6, \\
f_6 &= 1/3.
\end{align*}

The pair of two-joint antagonists (muscles 5 and 6) exert force simultaneously.

GENERAL COMMENTS

The example will now be extended to allow general joint moments in the case $\beta = 0.5$ and $\alpha = 2$. Solving Equations (15)–(18) explicitly for $f_1$, $f_4$, $f_5$, and $f_6$, we obtain

\begin{align*}
f_1 &= (2/3)a + (1/3)b, \\
f_4 &= (1/3)a + (1/2)b, \\
f_5 &= (1/3)a - (1/6)b, \\
f_6 &= b/3,
\end{align*}

(19)  
(20)  
(21)  
(22)

where $a$ is the magnitude of the (negative) moment at the proximal joint and $b$ is the magnitude of the (positive) moment at the distal joint. Since $f_i \geq 0$, the linear constraints on $a$ and $b$ allowing for cocontraction of muscles 5 and 6 can be identified (Figure 2, feasible area). For pairs of values of $a$ and $b$ from the feasible area shown in Figure 2, the corresponding solutions for $f_1$, $f_4$, $f_5$, and $f_6$ can be directly calculated using Equations (19)–(22). Forces $f_2$ and $f_3$ will be zero for these conditions.

It is of interest to study the behavior of the system at the limits of the feasible area. For example, when either $a > 0 = b$ or $a = (1/2)b > 0$, cocontraction between muscles 5 and 6 will cease to exist. For $b = 0$, the solution can be directly read off from Equations (19)–(22): $f_1 = (2/3)a$, $f_4 = (1/3)a$, $f_5 = (1/3)a$, and $f_6 = 0$. For $a = (1/2)b$, we obtain the result $f_1(4/3)a$, $f_4 = (4/3)a$, $f_5 = 0$, $f_6 = (2/3)a$. 
RESULTS AND DISCUSSION

It was shown analytically, using the static, nonlinear optimization approach introduced by Crowninshield and Brand [4], that cocontraction of pairs of one-joint antagonistic muscles is not possible. This result generalizes the findings of Hughes and Chaffin [12], who obtained the same solution for a single-degree-of-freedom, two-dimensional system, and further supports statements in the literature that were made without explicit analytical support (e.g., [18]). However, this finding is in direct contradiction with Figure 10 of Davy and Audu [6], who exhibited cocontraction of soleus and tibialis anterior muscles using a two-dimensional musculoskeletal model and the static optimization approach of Crowninshield and Brand [4]. Based on our analytical findings, we must conclude that this particular (numerical) result reported in the literature [6] is not correct.
Zajac and Gordon [18] made the following statement about coactivation of antagonistic muscles using static optimization approaches: "A minimal stress cost function [i.e., the type of cost function used in this study] cannot result in the coactivation of antagonistic muscles." This statement is correct for coactivation of pairs of one-joint antagonists, but it does not hold, in general, for coactivation of pairs of two-joint or multijoint antagonistic muscles, with the exception of the special case where moment arms of such pairs of muscles are the same for each joint these muscles cross. This last case, however, is so special that it is not of general interest, possibly with the exception of the mathematical result it gives. In fact, a case where two multijoint muscles cross the same joints, produce opposite moments at each joint they cross, and have equal magnitudes of their respective moment arms at each corresponding joint for any joint configuration probably does not exist in an actual intact biological system.

Figure 3 shows cocontraction of muscles 5 and 6 for the situation of $\alpha = 2$ and $0 \leq \beta \leq 1$, where $\alpha$ is the power of the cost function and $\beta$ represents the moment arm of muscle 6 about the proximal joint. The proximal and distal joint moments that had to be satisfied by the muscular forces were set to $M_1 = -1$ and $M_2 = 1$, respectively. This particular case was solved analytically for $\beta = 0.5$ (i.e., $f_5 = 1/6$, $f_6 = 1/3$). For $\beta = 1.0$, cocontraction between muscles 5 and 6 does not occur, since their moment arms are equal in magnitude at both joints they cross. For $\beta = 0$, muscle 6 becomes a one-joint muscle identical to muscle 4 (Figure 1). Therefore, simultaneous forces in muscles 5 and 6

![Graph](image-url)  
**Fig. 3.** Cocontraction of muscles 5 and 6 as a function of $\beta$, the moment arm of muscle 6 about the proximal joint. $\alpha = 2$; $M_1 = -1$; $M_2 = 1$. 
do not represent coactivation of a pair of antagonistic muscles. In this case, muscle 6 becomes a one-joint antagonist to muscle 4 with identical moment arm and physiological cross-sectional area, and therefore the forces predicted for muscles 4 and 6 are identical (result not shown).

Figure 4 shows the cost function values corresponding to the cases shown in Figure 3. The cost increases with decreasing cocontraction between muscles 5 and 6 and reaches a peak for $\beta = 1.0$, when cocontraction has ceased to exist. Initially, this result, which suggests that cocontraction of antagonistic muscles is cost efficient, may appear counterintuitive. However, realizing that from a cost function point of view it is important to involve as many muscles as possible in a given task in order to keep stress values in individual muscles low, and further realizing that muscle 6 may be involved to a greater extent if the moment it produces about the proximal joint is small (i.e., $\beta$ is small) because it is in the undesired direction, Figure 4 can be explained.

The fact that muscles 5 and 6 simultaneously become silent at $\beta = 1.0$ in the above example (Figure 3) is not a general result. Let us consider the same case as that shown above except that $\alpha = 3$ and $M_1 = -3$. The corresponding muscular forces 5 and 6 and cost function values are shown in Figures 5 and 6, respectively. The results shown here are similar to those shown in Figures 3 and 4 except that muscle 5 remains active when $\beta = 1.0$.

![Figure 4](image-url)  
Fig. 4. Cost function values as a function of $\beta$, the moment arm of muscle 6 about the proximal joint. $\alpha = 2; M_1 = -1; M_2 = 1$. 
The theoretical results described above have some intriguing physiological implications. If we assume that reducing mechanical stress in muscles for a given movement task is associated with a decreased metabolic cost, we may consider the cost function values as indicators of

Fig. 5. Cocontraction of muscles 5 and 6 as a function of $\beta$, the moment arm of muscle 6 about the proximal joint. $\alpha = 3; M_1 = -3; M_2 = 1$.

Fig. 6. Cost function values as a function of $\beta$, the moment arm of muscle 6 about the proximal joint. $\alpha = 3; M_1 = -3; M_2 = 1$. 
efficiency. Figures 4 and 6 show that the cost function values are sensitive to small changes in a single parameter (\( \beta \)) of the musculoskeletal system. Furthermore, Figures 3 and 5 indicate that force sharing between muscles (in this case muscles 5 and 6) is also sensitive to those changes. Therefore, it appears that musculoskeletal geometry in general, and the ratio of moment arms of two-joint or multijoint muscles specifically, may significantly influence efficiency of movement and force sharing between muscles. The results of this study also indicate that cocontraction of pairs of antagonistic muscles may be efficient in many situations. This is a surprising result that may not be intuitively obvious from a physiological or motor control point of view.

The model used here predicts cocontraction only for pairs of muscles spanning two or more joints. It has been observed experimentally that cocontraction exists for one-joint muscle pairs under certain conditions, and it would be interesting to develop modifications (e.g., types of additional constraints or dynamic situations) that predict the observed behavior theoretically. Another interesting modification of the present model is to three-dimensional models of hinge joints, where motion is almost planar. Qualitatively, our optimization results remain valid under small perturbations, and we hope to discuss the definition and existence of antagonism and cocontraction for almost planar systems in a subsequent study.

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