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RESEARCH ARTICLE

On rhythmic and discrete movements: reflections, definitions and implications for motor control

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Abstract At present, rhythmic and discrete movements are investigated by largely distinct research communities using different experimental paradigms and theoretical constructs. As these two classes of movements are tightly interlinked in everyday behavior, a common theoretical foundation spanning across these two types of movements would be valuable. Furthermore, it has been argued that these two movement types may constitute primitives for more complex behavior. The goal of this paper is to develop a rigorous taxonomic foundation that not only permits better communication between different research communities, but also helps in defining movement types in experimental design and thereby clarifies fundamental questions about primitives in motor control. We propose formal definitions for discrete and rhythmic movements, analyze some of their variants, and discuss the application of a smoothness measure to both types that enables quantification of discreteness and rhythmicity. Central to the definition of discrete movement is their separation by postures. Based on this intuitive definition, certain variants of rhythmic movement are indistinguishable from a sequence of discrete movements, reflecting an ongoing debate in the motor neuroscience literature. Conversely, there exist rhythmic movements that cannot be composed of a sequence of discrete movements. As such, this taxonomy may provide a language for studying more complex behaviors in a principled fashion.

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Introduction

A hallmark of biological systems is the astonishing variety and complexity of their behavior, ranging from archetypal patterns such as locomotion and manipulation to cultural skills such as elite athletic performance and dance. For many good reasons, research in motor control has tended to focus on selected and restricted parts of this repertoire, such as walking, standing, and reaching, far short of the complexity seen in everyday behavior. There also has been relatively little overlap in experimental paradigms, methods of analysis, and theoretical constructs. This seems like a missed opportunity. Examination of studies from different research communities using different paradigms may yield new insight; after all, there is only one nervous system.

One prominent example of compartmentalization in the research literature is the separation between rhythmic and discrete movements such as locomotion and reaching. These actions are studied by different research communities and different control mechanisms have been hypothesized. For example, central pattern generators that produce rhythmic actions in locomotion have been a major focus of neurophysiological studies of neuronal structures in the spinal cord (Delcomyn 1980; Grillner 1975; Marder and Calabrese 1996). A more theoretical perspective has proposed coupled oscillator models as constructs to account for synchronized bimanual rhythmic coordination (Haken et al. 1985; Amazeen et al. 1998; Sternad et al. 1992). Voluntary reaching in different directions has been the predominant paradigm for which motor cortical functions have been examined (e.g., Kurtzer et al. 2006; Naselaris et al. 2006a, b). Goal-directed reaching has also been a favorite paradigm of studies to explicate the function of the cortical and cerebellar structures in their role of instantiating postulated internal models (Prablanc et al. 2003; Shadmehr and Wise 2005; Vindras et al. 2005). Does this difference of explanatory constructs for rhythmic and discrete movements reflect different challenges of these behaviors or simply the theoretical presumptions of different investigators? Are discrete movements fundamentally distinct from rhythmic movements?

Remarkably few attempts have been made to encompass both types of behaviors in one experimental paradigm or one single theoretical approach. At the behavioral level, rhythmic and discrete performance has been addressed in the context of the Fitts paradigm. Differences of movement time and peak velocity in rhythmic (reciprocal) and discrete performance were reported for the same index of difficulty (Smits-Engelsman et al. 2002; Guiard 1993, 1997; van Mourik and Beek 2004); conversely, different indices of difficulty were shown to elicit either rhythmic or discrete performance (Buchanan et al. 2003, 2004, 2006). Pauses between movements (‘‘dwell’’ times) have also been shown to vary with index of difficulty and with movement speed (Adam and Paas 1996; Teeken et al. 1996; Winstein et al. 1995). Sternad and colleagues conducted a series of experiments on the interaction between discrete and rhythmic elements in one movement, specifically the initiation of a discrete movement against a background of rhythmic movement. This issue has been examined in unimanual and bimanual coordination, both in a single-joint and a multi-joint task (de Rugy and Sternad 2003; Dean 2001; Sternad et al. 1998; Sternad 2007; Sternad and Dean 2003; Sternad et al. 2000a, 2006; Wei et al. 2003). One result central to all task variations was that the initiation of the discrete component was constrained to a phase window of the ongoing rhythmic movement. A model with two pattern generators for this interaction has been formulated to account for experimental observations (de Rugy and Sternad 2003). Similar issues have been examined in the context of tremor (Elble et al. 1994; Wierzbicka et al. 1993).

In neurophysiological experiments, Mink and Thach (1991) aimed to elucidate the role of the basal ganglia in the performance of different modes of behavior, i.e., variations of rhythmic and discrete movements. They recorded single cells of the globus pallidus and dentate nucleus in monkeys performing five different tasks involving flexion and extension of the wrist. Variants of discrete and rhythmic movement tasks were accompanied by different neuronal discharge in the globus pallidus, while no such dissociation was found in the dentate nucleus or other behavioral and EMG variables. At a theoretical level, Schöner (1990) presented an analysis of a dynamic equation to demonstrate by example that a single dynamic system can display several parameter regimes, some of which produced limit cycle (rhythmic) behavior, others fixed point (discrete) behavior. Jirsa and Kelso (2005) proposed a topological approach that generates both types of behaviors and transitions between them.

With these few studies on both rhythmic and discrete actions, not only are neuronal correlates or theoretical approaches yet to be established, even the basic behavioral distinction between rhythmic and discrete movements is not always clear or consistent. Is arm-swinging rhythmic in the same way as walking? The latter includes intermittent hard contact with the ground which might evoke substantial differences. Indeed, some studies analyze locomotion as a sequence of individual steps rather than a continuous rhythmic movement (Winter 1990). Furthermore, frequency or pace may affect the difference between rhythmic and discrete movements. If performed sufficiently slowly, continuous cyclic arm movements exhibit kinematic fluctuations that suggest they may be executed as a sequence of discrete movements (Nagasaki 1991; Doeringer and Hogan 1998). Conversely, studies of discrete reaching have typically assumed it is a single, smooth action, yet again, sufficiently slow reaching movements exhibit kinematic fluctuations (Dipietro et al. 2004) which are especially evident in the movements of neurologically injured patients (Krebs et al. 1999). Furthermore, rhythmic and discrete actions are frequently intertwined: playing piano, we press the keys rhythmically and simultaneously translate the hand to reach target keys; walking, we routinely place the foot on a visually perceived target, adjusting the stride to reach this target; handwriting may be regarded as a rhythmic movement or as a sequence of discrete strokes strung together or both. Finally, intention tremor during a voluntary reach is a combination of rhythmic and discrete movements, although with different time scales and different degrees of functionality. Behaviorally, the relation between rhythmic and discrete movements appears complex.

A common assumption in motor control studies is that complex behavior is made up of elementary building-blocks, synergies or primitives, with concomitant implications for the neural structures that generate these primitives. Is there one primitive such that all behavior boils down to the concatenation or linking of basic control elements? Or is there more than one primitive, each subserved by different neural or control structures? Considering rhythmic and discrete movements, three possibilities have been proposed (Dean 2001; Sternad et al. 2000; Buchanan et al. 2003; Schaal et al. 2004): first, discrete movements are fundamental, with rhythmic movements being concatenations of discrete movements; second, rhythmic movements are fundamental, with discrete movements being truncated rhythmic movements; and third, rhythmic and discrete movements are two different and independent primitives.

In brain imaging studies, Sternad and colleagues have shown that different areas of the human brain are involved in the production of certain rhythmic and discrete movements (Schaal et al. 2004; Yu et al. 2007). Specifically, rhythmic movements required significantly less cortical and subcortical involvement than discrete movements. At least in those experiments, rhythmic movements cannot be a concatenation of discrete movements, though the possibility that discrete movements are truncated rhythmic movements remains open. Indirect support of this distinction also comes from Graziano's work which shows that stimulation of the motor cortex only elicits movements to final postures, i.e., discrete movements, and never rhythmic movements (Graziano et al. 2002). Behavioral support for the hypothesis of two different primitives has been reported in experiments on Fitts' Law (Buchanan et al. 2003; van Mourik and Beek 2004). Taken together, these studies suggest that comparison of discrete and rhythmic movements may facilitate discrimination between control mechanisms at different levels of the central nervous system.

But what exactly are rhythmic movements? And what are discrete movements? One difficulty is that the labels for the different actions under investigation are often defined in an ad hoc fashion, or motivated by common language with many different associations and connotations, or applied to different variants of movements, resulting in the confusing (sometimes contradictory) usage that reflects the compartmentalization of the motor control community. For example, recently Ivry and colleagues reported that discrete or intermittently rhythmic movements such as finger tapping require different neural substrates from continuously rhythmic movements such as continuous circle drawing; the cerebellum was argued to be the structure needed for explicit timing (Ivry and Spencer 2004; Spencer et al. 2003). Both actions are largely periodic but tapping includes contact events and evidently involves different brain regions. Are tapping movements rhythmic or discrete or do they represent an entirely different primitive element? However, in their use of the term, *discrete* tapping movements are performed in a *rhythmic* fashion, but each repetition is separated by a salient event. This use of the word *discrete* differs significantly from that used above. Until these differences are reconciled, it will be difficult to answer even basic questions, such as whether rhythmic movements are strings of discrete movements; discrete movements are truncated rhythmic movements; or the two are mutually exclusive classes.

We perceive that the growing science of motor control would benefit from a consistent definition and use of terms. Such a taxonomy of behavior should be consistent with our understanding of motor control to date but it should also clarify and potentially open up new research questions. In this paper we propose definitions of *rhythmic* and *discrete* and identify conditions under which these movement classes are distinct.

Approach

> ''*Everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions.*''
>
> Felix Klein

In our view, the only practical framework for such a taxonomy is mathematics. We recognize that the value of mathematical rigor may be questioned; many fields of science appear to progress satisfactorily in the absence of rigorous theory or formal definitions. Indeed, to some extent a lack of precision may facilitate the emergence of understanding, while premature attempts at rigor may seem constraining. However, an appropriate formulation of mathematics need not be confining. In particular, the definitions we propose include the inevitable presence of natural variability and measurement imprecision. Furthermore, as disciplines mature and understanding emerges, rigor and precision typically follow. It is our impression that movement science is reaching the stage at which concepts and ideas emerge with increasing rigor and precise definitions are appropriate.

In the following we attempt a definition of terms to describe rhythmic and discrete behavior. Our goals are to disambiguate the wealth of terms taken from common language and used to describe motor behavior, to clarify the scientific discourse, facilitate a better ordering of the extant literature, and provide a foundation to study the more complex actions found in behavior. We attempt to be consistent with the predominant use of terms but in the interest of internal consistency we may at times deviate from some usages. Inevitably, some of our definitions state the obvious but we believe this is not only an essential step but also advantageous as it subsequently enables subtler distinctions without becoming counterintuitive. As such, we also hope that our suggestions find better acceptance in the research community. Eventually, the goal is to disentangle complex behavior. We begin with the simplest candidate definitions, then refine them to reflect the subtleties and nuances of real behavior. For clarity we assume that observables are expressed as real-valued scalar functions of a real-valued scalar argument, usually time (e.g., limb position as a function of time). However, this does not limit the definitions to descriptions of kinematics.

Central in our attempt to clarify terminology is that we begin with the observable *product*, not the *process* that gives rise to it. In other words, we define terms used in the phenomenology of motor behavior, not in the hypothesized mechanisms that give rise to observable behavior. Researchers who study motor control are generally concerned to detail the neural processes that generate movement and behavior. However, it needs to be kept in mind that generative processes are inferred from observing their products which are the only data accessible to measurement, such as overt patterns of kinematics, forces or muscle activities. Given that the study of pertinent control mechanisms is ongoing, the following exposition intentionally refrains from any references or speculations about their control mechanisms. First, the description of the product needs to become unambiguous in both criteria and terminology.

Discrete movements

Reaching to grasp a glass or place an object at a location, reaching out to open a door or to shake hands are ubiquitous examples of point-to-point, goal-oriented or target-oriented movements. In keeping with frequent usage when compared with rhythmic behavior, we propose to use the term *discrete* to describe this class of movements. Common mathematical usage is instructive: discrete refers to ''... taking a succession of distinct values'' (Borowski and Borwein 1991). The central idea to be gleaned from this mathematical definition is ''distinct''. In order for two movements to be distinct, there must be a gap between them, an interval of no movement. That is, a discrete movement has an unambiguously identifiable start and stop; *discrete* movements are bounded by *distinct* postures.

Posture

In the context of movement, we propose that the terms ''stop'', ''pause'' and ''pose'' are all synonymous with ''posture'' which we define as the absence of movement.

Preliminary definition A fixed posture occupies a non-zero duration in which no movement occurs.

To be more precise, if a movement is described by a function $y(t)$, a fixed posture is defined by

$$y(t_i) = y(t_j) \text{ for all } t_i \text{ and } t_j \text{ in the interval}$$
$$p: t_p - \delta_p < t_i,\ t_j < t_p$$

where the duration of the posture is not zero, $\delta_p > 0$.[1] An important point is that the identical values of y during a non-zero interval mean that *all* derivatives of $y(t)$ are zero in this interval. In fact, this may be used as an equivalent, operational definition; thus a fixed posture is characterized by:

$$\mathrm{d}^n y(t) \big/ \mathrm{d}t^n = 0 \text{ for all } n = 1, 2, 3, \ldots$$
$$\text{and for all } t \text{ in the interval } p: t_p - \delta_p < t < t_p$$

For practical purposes, this preliminary definition is excessively strict. First of all, real measurements may be corrupted by artifactual noise, rendering it impossible to observe identical values of the function $y(t)$ over any non-zero interval. Conversely, due to the limited response time and resolution of practical instrumentation, measurements taken at sufficiently short intervals may appear identical even when movement never stops. Some account of measurement imprecision is needed to make this definition useful.

A second and perhaps more fundamental role of imprecision is to enable a formal definition of terms that might otherwise be undefinable or meaningless.[2] Fixed posture in the usual behavioral meaning of the term is not characterized by absolute stillness; for example, tremor is omnipresent and postural drift is common. Thus, even aside from measurement limitations, the absolute stillness implied by this preliminary definition has little behavioral meaning. We therefore propose a more general definition as follows:

Definition (revised) A posture occupies a non-negligible duration in which only negligible movement occurs.

To be more precise,

$$|y(t_i) - y(t_j)| < \varepsilon \text{ for all } t_i \text{ and } t_j \text{ in the interval}$$
$$p: t_p - \delta_p < t_i,\ t_j < t_p,$$

[1] This notation means that both t_i and t_j are contained within the interval between $t_p - \delta_p$ and t_p.

[2] A familiar example is the tangent to a curve, a line which touches the curve at a single point with the same slope as the curve at that point. Of course, the slope of a point is meaningless but the slope of a curve at a point may be defined by considering a line through two suitably close points and noting the limiting value to which its slope converges as the separation of the points diminishes. This avoids the ''divide-by-zero'' problem which would otherwise confound attempts at precision.

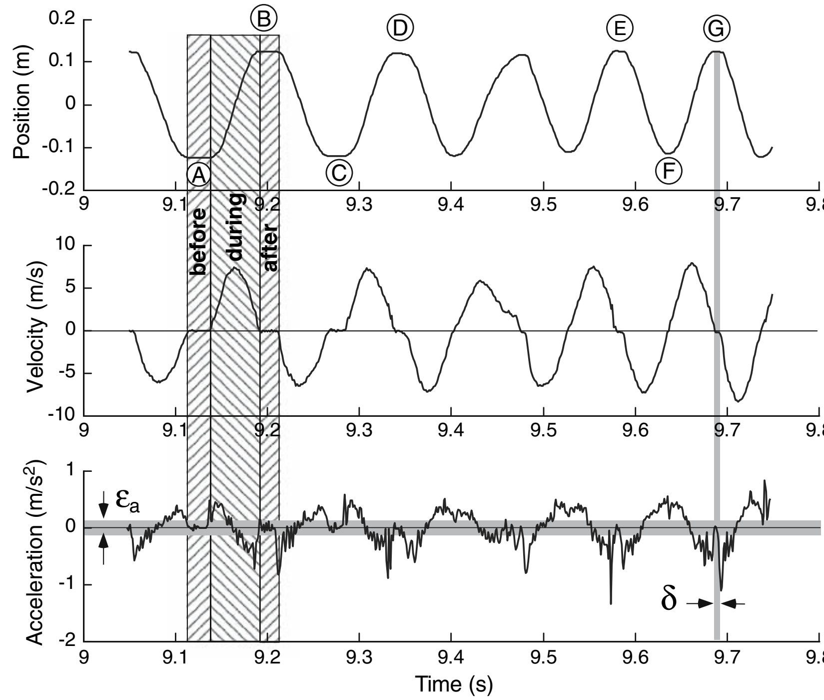

Fig. 1 A segment of an exemplary trajectory performed by a subject paced by an auditory metronome at ever-decreasing time intervals. The three *panels* depict the position, velocity and acceleration records, as found by numerical differentiation of the raw position data

where ε > 0 and $\delta_p > 0$ are small positive constants. The exact definition of ''small'' will depend on context and these constants will typically be used to minimize measurement artifact or to distinguish tremor from a departure from fixed posture.

Discrete movement defined

With a definition of posture we now define a discrete movement.

Definition A discrete movement is preceded and succeeded by postures and occupies a non-negligible duration containing no posture.

To be precise, there exist three adjacent, non-overlapping intervals of non-negligible duration in the domain of the function $y(t)$, which we may term *before* (b), *during* (d) and *after* (a), such that:

before is an interval b: $t_b - \delta_b < t_i,\ t_j < t_b$ in which $|y(t_i) - y(t_j)| < \varepsilon$ for all t_i and t_j,

where $\delta_b > 0$ and ε > 0 are small positive constants.

during is an interval d: $t_b \le t \le t_a$ containing no postures; there is no interval

$t_k < t_i,\ t_j < t_h$ with $|y(t_i) - y(t_j)| < \varepsilon$ for all t_i and t_j,

where $t_b \le t_k < t_a$, $t_b < t_h \le t_a$, $t_h = t_k + \delta_d$, and $\delta_d > 0$ and ε > 0 are small positive constants.

after is an interval a: $t_a < t_i,\ t_j < t_a + \delta_a$ in which $|y(t_i) - y(t_j)| < \varepsilon$ for all t_i and t_j, where $\delta_a > 0$ and ε > 0 are small positive constants.

While these first definitions of posture and discrete movement are consistent with intuition and common usage, an example may illustrate some of the subtleties in a more complex situation. Figure 1 presents a segment of kinematic data recorded from a human subject[3] who made a series of back-and-forth movements between two visually-presented targets in the horizontal plane. The top panel shows position (recorded with Ascension Technologies' Flock of Birds); the middle and bottom panels show velocity and acceleration, obtained by numerical differentiation. The subject followed audible cues to move between the two targets at ever-decreasing intervals. Examining the first region with minimum position (region A) it is clear from the position record that the subject has come to rest for a brief period (a ''dwell'' time) at the minimum position and the velocity and acceleration records confirm this observation. At the following maximum position (region B) the subject also comes to rest for a brief period. In between (during) is an interval of movement containing no rest. These are examples of the *before*, *during* and *after* intervals required to define a discrete movement.

Of course, the actual data records do not remain constant in the *before* and *after* intervals but fluctuate, partly due to measurement noise accentuated by differentiation. One use

[3] Subjects gave informed consent as approved by the Institutional Review Boards of the Massachusetts Institute of Technology and the Pennsylvania State University.

of the parameter ε is to define a threshold value that quantifies the resolution of the measurement system. In general, the resolution of each kinematic variable will differ and in this case, different values are required for position, ε_p, velocity, ε_v, and acceleration, ε_a.

A second (and behaviorally more important) use of ε is exemplified at region D in the figure. In this region the measurement system resolves a small, slow drift of the subject's position around its maximum value. The velocity and acceleration records indicate a clear break between adjacent movements (e.g., note the ''shoulder'' in the velocity profile at D) and on these grounds, this region marks the end of one discrete movement and the beginning of the next. However, the velocity and acceleration records also confirm that the limb does not remain exactly at rest in this region. A more subtle example is evident at region C. The acceleration and velocity records clearly fluctuate about zero but with larger amplitudes than at A. The important point here is that the posture between movements is not characterized by absolute stillness. To allow for this, the values of ε should be larger than those solely determined by measurement resolution.

The necessity of parameter δ is illustrated in the regions including E and F and G, where the postures that delimit discrete movements begin to disappear. At the maximum position values (E and G) the subject comes to rest at a posture as evidenced by velocity and acceleration simultaneously coming to zero (more correctly, to within the thresholds ε_v and ε_a, respectively). However, at the minimum position, F, the subject does not assume a posture, as the acceleration fluctuates about its local maximum while the velocity passes through zero. The briefest duration of the posture required to delimit a discrete movement may be identified objectively based on one representative instance. For example, δ may be assigned as the interval at G in which acceleration remains within the region defined by ε_a. Durations shorter than this value are (by definition) negligible. To illustrate, consider the brief fluctuations in the acceleration record near the minimum position at F. They may fall within the region defined by ε_a but they do not define an identifiable posture as they do not remain within that region for a period as long as δ.

Returning to the formalism, defining discrete movements as occurring between identifiable postures serves to emphasize the unique neural substrates that may be required for the control of discrete movements. Occupying and maintaining a posture may impose different demands than movement itself. For example, maintenance of posture requires stability about a particular body configuration, but it may or may not be required for movement. Thus, stopping at a specified posture may evoke different neural substrates and mechanisms than moving to or through the same body configuration. Gymnastics provides an intuitive example: Imagine a gymnast performs giant swings on the high bar (in which the extended body rotates about the bar). Going *through* the vertically upright position requires entirely different control than the challenge of coming to a *stop* at the handstand (vertically upright) position.

Rhythmic movements

The term *rhythmic* denotes a plethora of actions, ranging from the archetypal behavior of locomotion to more cultural forms of movements such as drumming. There are a number of different ''rhythmic'' movements: those that are continuous without interspersed breaks and others that involve a contact event or rest alternating with movement. Still others may be very slow and thereby lose their periodicity, or they may progressively speed up and therefore be non-periodic. In the timing and musical literature it has sometimes been stated that the essence of rhythm is its deviation from strict periodicity (Fraisse 1963). As the term *rhythmic* is ubiquitous and apparently has numerous instantiations and connotations, we suggest it should serve as a generic descriptor, an umbrella term for a class of movements, which will be differentiated into a number of sub-types as detailed below.

Periodic or cyclic movements

A common (though not universal) characteristic of rhythmic movement is its periodicity. The terms *periodic* and *cyclic* are often used interchangeably, and we suggest they should be regarded as synonymous. Though the term *periodic* has a strict mathematical definition, common usage is somewhat looser. For clarity we first define *strictly periodic*, then identify the main forms of approximately periodic behavior encountered in motor control studies.

Strictly periodic movements

A movement is (strictly) periodic if all of its values recur at regular intervals.[4] To be more precise, a function is strictly periodic when adding any integer multiple of a constant to its argument returns the same value of the function,

$$y(t) = y(t + nT) \text{ for all } t \text{ and for } n = \pm 0, 1, 2, \ldots$$

where T is a constant. The smallest value of the constant T is the period. Trigonometric functions provide common examples: the function

[4] It should be noted that because a strictly periodic function has infinite duration, periodicity can never be proven conclusively from experimental observation, and certainly in biology deviations from strict periodicity should be anticipated.

$$\sin\left(\frac{2\pi}{T}t\right) = \sin\left(\frac{2\pi}{T}(t+nT)\right)$$
$$\text{for all } t \text{ and for } n = \pm 0, 1, 2, \ldots$$

is periodic with period T.

It is useful to distinguish two common types of approximately periodic or cyclic movements which we propose to term *almost periodic* and *transiently periodic* movements, respectively.

Almost periodic movements

This term describes a movement that remains close to periodic for indefinitely long intervals. Examples are ubiquitous: almost all biological rhythms, ranging from heartbeat to rhythmic finger tapping and running, exhibit fluctuations centered on an identifiably periodic behavior. Addition of noise or random fluctuations to a periodic function may render it non-periodic but note that not all small departures from periodicity are necessarily random or stochastic. An alternative is weakly chaotic behavior, displayed by certain nonlinear deterministic dynamical systems; it is close to periodic, yet never returns to the same values. Even though chaos remains difficult to identify conclusively, fluctuations in biological data have been used as a window to understand the processes giving rise to the observed rhythmic movement (Dingwell et al. 2001; Hausdorff et al. 2001; Wing and Kristofferson 1973).

We suggest the following definition for this class of deviations from strict periodicity:

Definition A movement is *almost periodic* if all of its values approximately recur at approximately regular intervals.

To be more precise, a function is almost periodic if

$$\left|y(t_i) - y(t_j + nT)\right| < \varepsilon \text{ for all } t_i \text{ and } t_j$$
$$\text{such that } \left|t_i - t_j\right| < \delta \text{ and for all } n = \pm 0, 1, 2, \ldots$$

where T is the period and the constants ε and δ are small in some suitably-defined sense, for example

$$\varepsilon \ll (\max(y) - \min(y)) \text{ and } \delta \ll T.$$

The idea we intend to capture is that the *average* time course of an almost periodic movement is strictly periodic. Denoting the sets of periods and amplitudes of different cycles by $\{T_i\}$ and $\{A_i\}$ respectively, for an almost periodic function,

$$\left|T_i - T_j\right| \ll T \text{ for all } i,j \text{ and } \left|A_i - A_j\right| \text{ for all } i,j.$$

As a result the individual periods and amplitudes are clustered around the representative values T and A.

Quasi-periodic movements

Almost-periodic movements as defined above should not be confused with *quasi-periodic* movements, a different departure from periodicity. Concurrent rhythmic movements of the body may give rise to trajectories that may or may not be periodic. For example, trajectories of head movements may result from a combination of the rhythmic movements of locomotion and breathing. Even if these components are periodic but the ratio of the component periods is not a rational number, the head trajectory will not be periodic. This reflects an important subtlety of the mathematics: a sum of periodic functions need not be periodic. Consider the sum of two sinusoids

$$y(t) = A_1 \sin\left(\frac{2\pi}{T_1}t\right) + A_2 \sin\left(\frac{2\pi}{T_2}t\right),$$

where the amplitudes A_1 and A_2 and the periods T_1 and T_2 are constants. This function is periodic if and only if the period T_1 is a rational multiple of the period T_2. Nevertheless, the function clearly exhibits a quantifiable temporal regularity; for example, its Fourier spectrum will exhibit two clearly defined peaks corresponding to the two sinusoidal components. This phenomenon is termed quasi-periodicity. For our present purpose there is no need to adapt the standard definition and we refer the reader to a text such as Strogatz (1994) for details and further discussion.

Transiently periodic movements

A conceptually different type of deviation is a progressive or systematic departure from periodicity. This may be a progressive variation of amplitude or period or both. A lightly-plucked string oscillates in a form that may be described by

$$y(t) = Ae^{-bt}\sin(\omega t),$$

where A, b and ω are positive constants. This function is not periodic because it has continuously changing amplitudes, although it crosses zero at strictly periodic intervals. Another example is playing piano, where *accelerando* refers to a systematic speeding up of tempo, *ritardando* to a systematic slowing down. An example from human behavior is a sprinter leaving the starting blocks who shows increasing stride lengths in the first accelerating part of the sprint; the period may or may not vary accordingly. To render precision, we propose the following definition:

Definition A movement is *transiently periodic* if large differences may occur between cycles but the differences between *adjacent* cycles are small.

In mathematical terms, given ordered sets of the periods $\{T_i\}$ and amplitudes $\{A_i\}$ of successive cycles, a function is transiently periodic if

$$\left|T_i - T_{i+1}\right| \ll T_i \text{ for all } i \quad \text{or } \left|A_i - A_{i+1}\right| \ll A_i \text{ for all } i \text{ or both.}$$

Note, however, that the differences between nonadjacent cycles may be arbitrarily large.

An example is evident in the kinematic record of Fig. 1. The movement displays temporal regularity though it is clearly not periodic. Between about 9.05 and 9.3 s adjacent cycles are approximately similar; again, between about 9.5 and 9.7 s adjacent cycles are approximately similar; however, the cycle between E and G is markedly different from the cycle between B and D. Thus, any appearance of approximate periodicity is transient at most; this is an example of *transiently periodic* behavior.

Returning to the formalism, transiently-periodic describes a movement that deviates systematically from periodicity with a time course that is long compared to the period. As a result, any average amplitude or period is unlikely to be representative. If an average amplitude (or period) is computed over any two non-overlapping sets of adjacent cycles, these averages need not be equal, nor even close. A notorious failure of novice musicians when they first play in an ensemble is a tendency to increase tempo, an example of transiently periodic behavior.

Recurrent or repetitive movements

Finally, we remove the requirement for any temporal regularity. The most general characteristic of rhythmic movements is that some aspect is repeated. A movement is *recurrent* (or, synonymously, *repetitive* or *reciprocal*) if values of the function y are equal for different values of the argument,

$$y(t_i) = y(t_j) \text{ for } t_i \neq t_j$$

where t_i and t_j are members of a set of distinct values of the argument, $\{t_1, t_2, \ldots\}$. However, successive identical values of y need not follow at equal intervals. For example, in speech production gestural configurations may be recurrent even though their timing is not. As this defines the least restrictive class of rhythmic movements, we suggest that *recurrent* or *repetitive* should also allow a degree of imprecision and refine the definition as follows:

Definition A movement is *recurrent* if approximately equal values of the function correspond to a large number of significantly different values of the argument.

A formal rendition of this definition is lengthy and is presented in Appendix 1. Usually the repeated quantity will be a location revisited at different times. For example, drawing a rectangle in space where the trajectories traversing each side may have unequal duration is a repetitive behavior because each of the four corners is revisited repeatedly. Of course, features of behavior other than location may be repeated. For example, in handwriting, kinematic features such as particular values of curvature at loops or cusps may recur irregularly. Thumbing through a magazine, the forces applied to the pages may recur but at irregular intervals. A related application of the concept of recurrence can be found in time-series analysis of nonlinear dynamical systems: whether and how a system revisits locations in its state space, independent of when that happens, may be used as a sensitive indicator of stability. Several variants of such analyses have been developed (Casadagli 1997; Eckmann et al. 1987; Kantz and Schreiber 1997).

Rhythmic episodes and discrete sequences

One goal of our formal approach is to enable unambiguous classification of movements as rhythmic or discrete. Are the above definitions sufficient to clarify some of the more complex cases encountered in experimental studies and real behaviors? Have we succeeded and will our definitions aid in clarifying the discourse between researchers with different paradigms? To evaluate success we examine two extreme cases: Is a back-and-forth movement that begins and ends with postures distinguishable from a discrete movement? And is a rhythmic movement distinguishable from a sequence of discrete movements?

To address the first case, we re-examine the definition of discrete movements: while discrete movements are defined by intervals of no movement at their beginning and end, the form of the movement between the ends is not specified. In fact, an episode of transiently periodic behavior that started from an identifiable posture, moved back and forth for several cycles without stopping, and terminated at an identifiable posture would satisfy the proposed definition of a discrete movement. Thus an episode of rhythmic movement may be categorized as discrete.

Conversely, to address the second case, our definitions of rhythmic movements do not specify their detailed form. They must at least be recurrent and perhaps exhibit some time structure such as (almost) periodicity; otherwise any profile is permitted. Hence a sequence of discrete movements may be recurrent or even periodic. Thus, discrete movement sequences may belong to one or more of the sub-classes of rhythmic movements.

This ambiguity arises inevitably from the rich repertoire of natural behavior. By our definitions, non-recurrent movements between postures are unambiguously discrete, and recurrent movements without stops are unambiguously rhythmic. However, like many linguistic categories, the two classes *rhythmic* and *discrete* are not mutually exclusive and they do not have exact boundaries. Rather, there is a continuum or ''grey area'' between unambiguously rhythmic and unambiguously discrete movements. Such lack of sharp demarcation is not unusual and considerations like these prompted the development of fuzzy logic as an extension of predicate logic (Zadeh 1965). To clarify membership additional considerations that go beyond the definitions proposed above are required. In the following we show that one such consideration is smoothness. Defining a measure of smoothness that is applicable to both rhythmic and discrete movements, we show that it may provide a quantitative measure to establish a degree of discreteness and a degree of rhythmicity. We will demonstrate this for the two transitional cases listed above and for the experimental data in the regions E–F–G of Fig. 1.

Smoothness

In mathematics, differentiable functions are often referred to as smooth, where these two adjectives are used more or less interchangeably. A function is *continuous* if it has no abrupt changes[5] and *differentiable* if its derivative exists. However, in the context of movement it is unsatisfactory to identify smoothness with differentiability because differentiability does not preclude extremely rapid rates of change. In common usage smoothness is a descriptor for any movement that is not ''jerky''. For example, typically the velocity of a sequence of saccadic eye movements may be described as jerky or even discontinuous. A closer examination shows that because the eye has non-zero inertia and eye-muscles cannot generate infinite force, velocity is, in fact, continuous and differentiable. In fact, any one saccade has a bell-shaped velocity profile. Nevertheless, the sequence of movements is not smooth.

A measure of smoothness

Smoothness is a matter of degree, not a categorical distinction, but by identifying the smoothest rhythmic and discrete movements we will provide a quantitative benchmark for each of these movement classes. To proceed, we define a measure of smoothness that assigns a scalar to possible movements. The smoothest movement minimizes that scalar measure. One measure of smoothness that has proven useful in motor control studies is mean-squared jerk, *msj*, the rate of change of acceleration or the third time-derivative of position. Accordingly, we use the following definition of the smoothest movement:

Definition The smoothest movement between y_1 at t_1 and y_2 at t_2 is that which minimizes the scalar

$$msj = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{2} \left| \mathrm{d}^3 y / \mathrm{d}t^3 \right|^2 \mathrm{d}t.$$

The *boundary conditions* (y_1 at t_1 and y_2 at t_2) are an important part of this definition (Hogan 1982, 1984).[6] A movement with zero displacement has minimal-magnitude derivatives, because in this case all derivatives (including the ''zeroth-order'' derivative) are zero, but this trivial case has little value. Similarly, an infinitely slow motion would result in derivatives of minimal magnitude but this is another trivial case of little value; finite movement duration is also required.

The smoothest discrete movement

The *msj* smoothness measure may be applied to all movements and, as it is a matter of degree, may be used to rank them. Examples of discrete movements that are not maximally smooth are commonplace. Imagine a Parkinson patient's reaching movement: while pointing the hand from initial to target position, the trajectory is not smooth because of the pronounced tremor. A more intricate example is cursive handwriting. Strokes between landmark locations of the letters are typically curved and have changing velocity, thereby producing the individual ''signature'' in curvature and accentuation. By the proposed definition, they are not the smoothest possible movements; here smoothness yields in favor of some other characteristic such as individuality.

Using the *msj* measure, the smoothest discrete movement has the form of a fifth-order polynomial in time (Flash and Hogan 1985; Hogan 1982, 1984); see Appendix 2. Though several related alternative measures have also been

[5] A function $y(t)$ is *continuous* if at every point a in its domain, there exists a constant $\delta > 0$ corresponding to every constant $\varepsilon > 0$ such that $|y(t) - y(a)| < \varepsilon$ for all t in the neighborhood $|t - a| < \delta$. To be continuous at a *specific* point a the function must be defined at $t = a$ and $\lim_{t \to a} y(t) = y(a)$; a continuous *function* is continuous at all points in its domain.

[6] This is similar to the definition of familiar concepts such as the shortest line. The shortest line has zero length (a trivial answer), so we need boundary conditions, i.e., two points. The shortest line between two points may be found by defining a measure of length, which assigns a scalar to each possible path. This scalar is defined by adding all of the infinitesimal displacements along the path. Variational calculus may then be applied to find the path with minimal length (a straight line in Euclidean space, a Great Circle on the surface of a sphere and so on). Note that length, like smoothness, is not a categorical distinction but a matter of degree.

proposed (Flash et al. 2003) the minimum *msj* movement provides one competent description of normal reaching that can serve as an archetypal discrete movement.

The smoothest rhythmic movement

Smoothness is also a hallmark of many rhythmic behaviors and *msj* may also be applied as a measure of rhythmic smoothness and used to rank-order rhythmic movements. Examples of rhythmic movements that are not maximally smooth are commonplace. When tapping the fingers on a table or rhythmically bouncing a ball with a racket aspects of the movement (such as the velocity of the fingertip or ball) change abruptly at the moments of impact (Sternad et al. 2000b, 2001). At these times the higher-order movement derivatives become large (infinite in the ideal limit of instantaneous impact) hence these are not the smoothest rhythmic movements. Similarly, in rhythmic locomotion, the trajectory of the foot between swing and stance is continuous but on contacting the ground the foot makes an abrupt, effectively instantaneous, transition between motion and rest, resulting in large higher-order derivatives.

Applying the *msj* measure to rhythmic movements requires care. For a periodic movement the measure fluctuates as the interval of measurement increases to include multiple cycles or parts of cycles. However, as the number of cycles included increases without limit, the measure converges to the value obtained over a single cycle. In non-periodic cases, the measurement interval should be taken between times at which the behavior recurs. From our definitions above, it will always be possible to identify these times because recurrence (even without any temporal structure) is the minimum requirement for a behavior to be considered rhythmic.

The *msj* measure can be used to define the smoothest rhythmic movement (Nelson 1983). As detailed in Appendix 2, the minimum *msj* movement is strictly periodic and essentially sinusoidal. Several related alternative measures have been proposed (Richardson and Flash 2002).

Disambiguating rhythmic and discrete movements

We showed above that our formal definitions of *discrete* and *rhythmic* (which essentially state the obvious, in keeping with intuitive usage) define overlapping classes. While some movements are unambiguously discrete (e.g., those with no reversals between stops) and some are unambiguously rhythmic (e.g., recurrent movements with no stops) there are also intermediate cases for which additional considerations are required to clarify membership. In the following we illustrate how a measure of smoothness (e.g., *msj*) may be used to quantify both rhythmicity and discreteness and apply it to movements that are neither unambiguously discrete nor rhythmic.

Figure 2 shows the two exemplary cases mentioned above that cannot clearly be assigned as either rhythmic or discrete: in the left column back-and-forth movements that begin and end with postures are compared with the smoothest discrete movement; in the right column the smoothest cyclic movement is compared with sequences of discrete movements. The left column shows three panels with the position, velocity and acceleration profiles of four maximally-smooth movements of equal duration that begin and end with postures: at time zero, position, velocity and higher derivatives are zero; at unit time, position is unity but velocity and higher derivatives are zero. However, three of the four movements are also required to pass through intermediate locations before reaching the final posture. In movement (i) the intermediate locations are such that the movement first travels from the start to the end location but then returns to the start before finally traveling to and remaining at the end. At the intermediate locations (highlighted by the circles on the position profile, panel A) the velocity is zero as the movement reverses (see panel B) but acceleration is not zero; in fact, it reaches a local extremum (see panel C). As all derivatives are not simultaneously zero, these intermediate locations are not postures delineating discrete movements. Because, according to our definition, the form of the trajectory between postures is unspecified (other than that it may contain no postures) movement (i) is a discrete movement of unit duration. However, movement (i) may also be considered a bout or episode of rhythmic behavior that visits two locations, albeit with a small number of cycles. Is it properly classed as rhythmic or discrete?

Evaluating *msj* for these movements provides a measure that differentiates between them. Compared with a maximally-smooth discrete movement of the same duration between the same locations—movement (iv)—the *msj* of movement (i) is 238 times larger. This arises because the reversals in movement (i) require larger values (on average) of all derivatives, including jerk. Movements (ii) and (iii) are similar to movement (i) but the intermediate locations are positioned partway between the start and end locations. This results in lower *msj* values but they are still much larger than that of the maximally-smooth discrete movement (by factors of 111 and 25, respectively). Consequently, even these cases (which might be considered marginal examples of episodes of rhythmic movement) are clearly distinguishable from a sufficiently smooth discrete movement. Note that this distinction is a matter of degree, not category; we do *not* propose a specific value of *msj* to serve as a demarcation or ‘‘cut-off’’ separating discrete and rhythmic. Nevertheless, we can state definitively that there

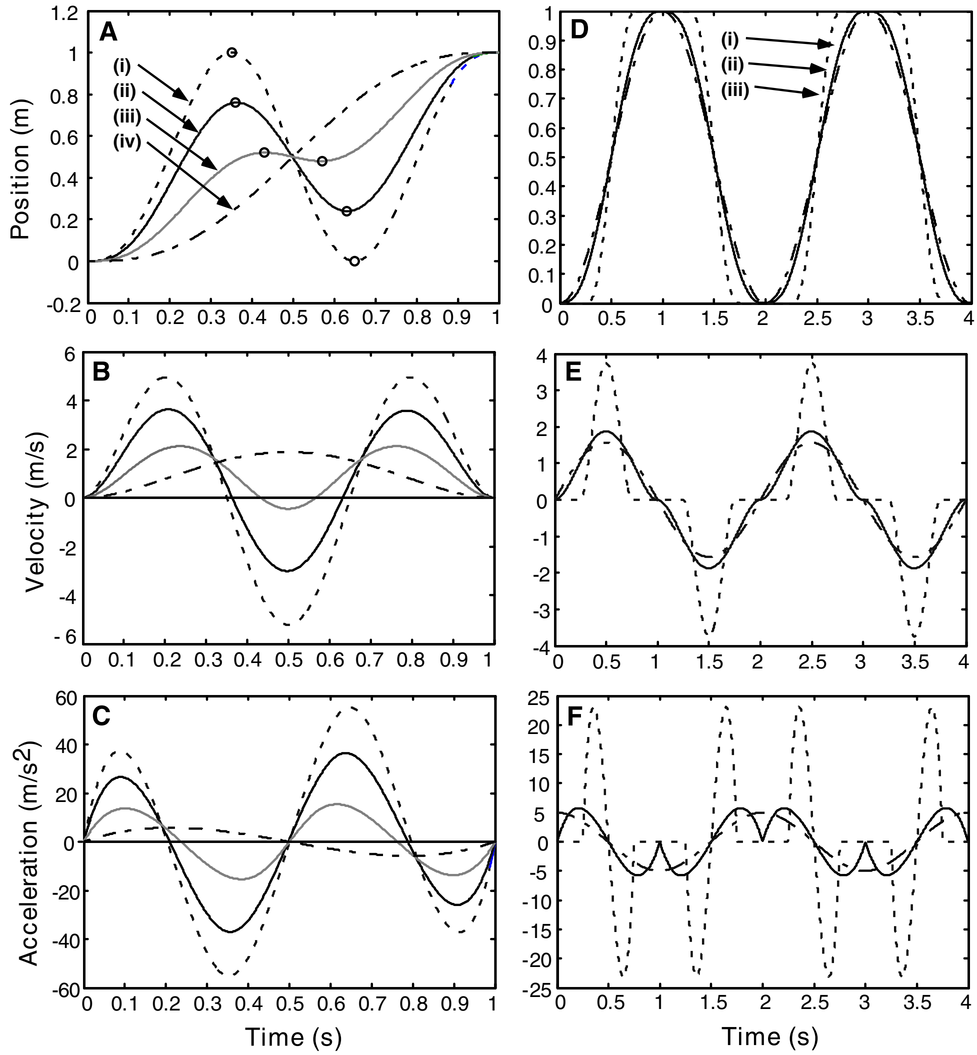

Fig. 2 Simulated data illustrating two test cases of variants of rhythmic and discrete movements. *Panels A*, *B*, and *C* show four movements exemplifying discrete behavior, although of different degrees of smoothness. *Panels D*, *E*, and *F* illustrate three trajectories with recurrence. While two movements are concatenations of discrete movements, one trace is the maximally smooth behavior, approximating a sinusoid

exist discrete movements (those with no reversals) that *cannot* be described as rhythmic.

The right column of Fig. 2 illustrates how smoothness may also distinguish a rhythmic movement from a sequence of discrete movements. Three panels show the position, velocity and acceleration profiles of two cycles of movements that repeatedly visit two locations. Movement (i) (dash-dot line) shows the maximally smooth rhythmic movement. Because it is only necessary to visit the locations that define the movement amplitude (but not to stop) the acceleration is not zero but reaches an extremum at these points (see panel F). In contrast, movement (ii) (solid line) shows a back-to-back sequence of discrete movements between the same two locations, each occupying the same duration as one half-cycle of movement (i). Being a sequence of discrete movements, the trajectory comes to a stop at the extreme positions. Is this movement properly classed as rhythmic or discrete?

Movement (ii) is a theoretical limit obtained when the duration of the stops required to delineate discrete movements approaches zero. Nevertheless the profiles of the derivatives (panels E and F) are markedly different from those of the maximally smooth cyclic movement (i); in particular, the acceleration profiles come to zero when displacement reaches an extremum. Both movements (i) and (ii) appear visually smooth (see panel D); both have continuous and differentiable displacement profiles. However, the *msj* per cycle of the sequence of discrete movements is six times greater than that of the maximally smooth cyclic movement. If a sequence of discrete movements includes stops of finite duration the contrast becomes even greater. In movement (iii) (dotted line) the stops last as long as the transitions between extrema and the *msj* per cycle is 192 times greater than that of the maximally smooth cyclic movement. The important point is that even the smoothest sequence of discrete movements is clearly distinguishable from the smoothest cyclic movement. Again, this distinction between rhythmic and discrete is a matter of degree, not category. Without proposing any ‘‘cut-off’’ value of *msj* to demarcate a boundary, we can state definitively that there exist some rhythmic movements (those without stops) that *cannot* be described as a sequence of back-to-back discrete movements. In between these two extremes (unambiguously rhythmic and unambiguously discrete) the two classes shade into each other.

Harmonicity

In the motor control literature several studies of rhythmic movements have introduced *harmonicity* as a measure of the degree of rhythmicity. Guiard (1993, 1997) defined an index of harmonicity, *H*, as the ratio of the highest and lowest absolute values of the local extrema of acceleration in each half-cycle. Harmonicity may also be measured using all of the available data (rather than selected landmarks) by the root-mean-squared deviation between a given trajectory and the best-fit simple harmonic motion (Sternad et al. 1999).

Considering Fig. 2, the smoothest rhythmic movement in the minimum *msj* sense has a normalized harmonicity of 99.8%; that is, the root-mean-squared deviation from a simple harmonic motion is less than 0.2% of the root-mean-squared value of a simple harmonic motion. For comparison, the smoothest sequence of discrete movements has a normalized harmonicity of 92.0% and the sequence of discrete movements with stops has a normalized harmonicity of 67.5%. Harmonicity may therefore also be used to distinguish rhythmic movements from a sequence of back-to-back discrete movements. Note, however, that it may not be applied to a single discrete movement. Given our goal of spanning rhythmic and discrete movements, their possible combinations and the transitions between them, a measure such as *msj* is required that may be applied to both. As we mentioned above, *msj* is not the only possible candidate. Other quantifications of smoothness are possible and may be superior. Further, wavelet analysis and other more sophisticated approaches may serve the same purpose. However, we see value in the simplicity and general familiarity of the research community with a measure such as *msj*.

Relation between theory and experiment

The features that distinguish rhythmic from discrete movements are readily observed in behavioral data. Comparing theory and observation, regions A, B and C in Fig. 1 exhibit position, velocity and acceleration profiles that (aside from measurement noise) resemble movement (iii) in the right column of Fig. 2, the sequence of discrete movements with interspersed stops. The movements through the maximum position at E and G in Fig. 1 exhibit kinematic profiles resembling movement (ii) in the right column of Fig. 2, the sequence of discrete movements without interspersed stops. Interestingly, the kinematic record of the movement through the minimum position at F shows no evidence of stopping in a posture. This illustrates that while the postures that delineate a discrete movement (e.g., at E and G) must be distinct, they need not be at different locations.

Discussion

The quest for primitives in movement generation

A prominent theme of motor control research is that complex movements are produced by combining elements from fundamental classes of primitive movements. Identifying and distinguishing these primitives or elementary movements is therefore a central challenge of many lines of investigation. Given the fundamental nature of this question a number of suggestions have been made: Inferring from animal experiments Giszter and colleagues have suggested force fields generated by spinal cord compartments serve as primitives (Bizzi et al. 1991; Giszter et al. 1993; Hart and Giszter 2004). Relatedly, synergies have been proposed by numerous researchers as inferred from different levels of analyses (d'Avella and Bizzi 2005; Latash 2005; Ting and Macpherson 2005).

Partly theoretically motivated, partly supported by brain imaging results Sternad and colleagues proposed that rhythmic and discrete movements are controlled by fundamentally different regions of the brain (Schaal et al. 2004; Yu et al. 2007). Ivry and colleagues have revealed differential involvement of the cerebellum for rhythmic movements with or without discontinuities (Ivry and Spencer 2004; Ivry et al. 2002; Spencer et al. 2003). Schmidt and Lee (2005) distinguish continuous, serial and discrete movements on the basis of observable kinematics. These lines of research focus on different levels in the motor control system, from neural activity in the cerebrum to muscle patterns to overt mechanical behavior. However, from the extant literature it is unclear how—or even whether—these primitives may be distinguished from one another.

The main goal of the work presented here was to disambiguate terminology so as to clarify the discussion of these questions. A related motivation was a perception that an unresolved debate, such as the relation between rhythmic and discrete movements, often indicates that closer examination may be productive. Are rhythmic movements governed by different control structures than discrete movements as the division of the literature implies? Or should these apparently different phenomena ultimately be produced by the same parts of the nervous system, albeit with different nuances?

Organizing concepts in formal terms helps to clarify them and provides a basis for comparison. It is important to emphasize again that the discussion of the observable product should not be confounded with the numerous processes or mechanisms that may give rise to it. Therefore the definitions describe only the observable behavior (Table 1).

In addition to proposing a rigorous definition of terms, we discussed a quantitative method to clarify the distinction between discrete and rhythmic movements and to address transitional cases. Using any measure of smoothness based on minimizing time derivatives, a move between postures that included multiple reversals or repeatedly visited similar positions is substantially less smooth than the typical transition between postures with a single-peaked speed profile. Thus, sufficiently smooth discrete movements (e.g., typical of unimpaired reaching behavior) may unambiguously be distinguished from an episode of rhythmic movement. Conversely, any recurrent and/or periodic behavior that included intervals of no movement—however brief—would be substantially less smooth than a rhythmic movement that never came to rest. A sufficiently smooth rhythmic movement (e.g., swinging the arm, nodding the head, shaking a paw) may unambiguously be distinguished from a sequence of discrete movements. In sum, the two major classes of movement may be regarded as partially overlapping sets. Though some rhythmic behaviors cannot be confused with discrete movements, and vice-versa, some movements may have both rhythmic and discrete features. The *msj* measure may quantify both a degree of discreteness and a degree of rhythmicity.

Rhythmic and discrete movements: different primitives?

One outcome of our analysis is that there are rhythmic movements that cannot be described as a sequence of discrete movements. If our definitions are accepted, the first of the three possibilities itemized in the introduction may be ruled out. This conclusion is consistent with inferences drawn from empirical behavioral studies (Guiard 1993; Buchanan et al. 2003; Smits-Engelsman et al. 2002; van Mourik and Beek 2004) but it is derived by analyzing the consequences of formal definitions based on common intuitive usage. The theoretical finding also parallels recent brain-imaging studies (Schaal et al. 2004; Lewis and Miall 2003; Yu et al. 2007) showing that discrete movements engage far more neural structures than rhythmic movements. Neurally, behaviorally, and theoretically, some rhythmic movements cannot be concatenations of back-to-back discrete movements.

Natural behaviors: superimposition and sequences

For clarity our discussion has assumed that an identifiable limb trajectory provides an unambiguous expression of task performance. In practice, it may be a challenge to identify precisely which features of a complex behavior are rhythmic or discrete. Thus, in expressive dance, no single anatomical landmark may be satisfactory as the primary effector used to express rhythm may change continuously. Similarly, in Western drumming, no single instrument reliably ''carries the beat'', and it may be expressed on a cymbal, a snare drum, a bass drum, or all of the above. However, even in these more complex behaviors, we submit that the same definitions should be applied to whatever combination of measurables serves as the focus of control.

We do not expect rhythmic and discrete movements in their pure forms to exhaust all the possibilities of natural behavior. However, a benefit of formal definitions is that they may provide a basis to describe more complex actions and how they change with practice. Trajectories will commonly reflect the influence of simultaneous and possibly independent rhythmic and discrete processes, e.g., a discrete reaching movement of the hand may be superimposed onto simultaneous rhythmic movements of locomotion. Conversely, rhythmic and discrete elements may be intertwined (Adamovich et al. 1994; Sternad and Dean 2003).

Table 1 Summary of definitions

Posture	Bodily configuration defined by a period of no movement
Discrete	Movement that is bounded by identifiable postures
Rhythmic	Generic class of behaviors with several sub-types
Strictly periodic	Movement that satisfies the common mathematical definition of periodicity
Almost periodic	Movement that deviates from strict periodicity but remains periodic on average
Transiently periodic	Movement that is close to periodic over adjacent cycles but may be far from periodic in the long run, deviating substantially in either amplitude or period or both
Quasi-periodic	Combination of periodic behaviors that may or may not be transiently periodic
Recurrent, repetitive or reciprocal	Movements with recurring configurations but not necessarily any temporal structure

Walking also frequently integrates the reaching action of the foot, when stepping over an obstacle (de Rugy et al. 2002). Further, natural behavior often manifests itself as a sequence of actions where each action may be a discrete movement, a rhythmic movement, or a combination of both. For example, cursive handwriting may be generated by modulating a rhythmic oscillation, interjecting discrete movements or superimposing the two (Hollerbach 1981). With practice, it is eminently plausible that a recurrent sequence of discrete movements, i.e., with no temporal regularity, may acquire temporal regularity and gradually become more smoothly rhythmic. Using the *msj* measure, Nagasaki reported that cyclic forearm movements at lower frequencies resembled a sequence of discrete movements while at higher frequencies they converged on the maximally smooth, symmetric speed profiles (Nagasaki 1991).

Indeed, there are many other natural behaviors that pose interesting challenges for future investigation: rhythmic movement that is passively constrained to stop from time to time due to task mechanics like hitting a drum; a recurrent movement in which the mechanics impose a low-pass filter such as trampolining; rapid movements with via points such as steering a car around a series of s-curves; or interaction with vibrating devices. The definitions we propose attempt to clarify two of the fundamentals, rhythmic and discrete movements, and provide tools to analyze more complex behaviors and uncover their deeper structure.

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Appendix 1: Definition of recurrent

To precisely define ''approximately equal'' and ''significantly different'', consider a particular value in the range of the function and denote the corresponding argument by t_i. Let t_{i+} denote the closest larger value of the argument at which the function differs by a small constant ε from that value,

$$t_{i+} = t_i + \delta_{i+}$$

where δ_{i+} is the maximum positive constant such that

$$\left| y(t) - y(t_i) \right| < \varepsilon \text{ for all } t \text{ in the interval } t_i \le t \le t_i + \delta_{i+}$$

and let t_{i-} denote the closest smaller value,

$$t_{i-} = t_i - \delta_{i-}$$

where δ_{i-} is the maximum positive constant such that

$$\left| y(t) - y(t_i) \right| < \varepsilon \text{ for all } t \text{ in the interval } t_i - \delta_{i-} \le t \le t_i.$$

The function is recurrent if there exists a set of arguments $\{t_j\}$ for which

$$\left| y(t_i) - y(t_j) \right| < \varepsilon \text{ and } t_j < t_{i-} \text{ or } t_j > t_{i+}.$$

This might be termed *minimally recurrent* as it may be satisfied by as few as two values of the argument. To precisely define ''a large number'' of recurrences, we require the number to grow without bound if the observation interval grows without bound. For any candidate t_j identified above, find t_{j+} and t_{j-} as above to identify the interval in which the function remains approximately equal and assign all t in the interval $t_{j-} \le t \le t_{j+}$ as one occurrence. Let N denote the number of members of the set $\{t_j\}$ and D denote the interval of observation, $0 \le t \le D$. Then the function is *indefinitely recurrent* if $\lim_{D\to\infty} N = \infty$. White noise is a theoretical extreme case which might be termed *infinitely recurrent*; in a finite interval of observation, all values in the amplitude distribution recur an infinite number of times with probability approaching unity.

Appendix 2: Smoothest discrete and cyclic movements

Using mean-squared-jerk as a measure, the problem of identifying the smoothest movement may be formulated using optimization theory as that of finding the function $y(t)$ that minimizes the scalar

$$msj = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \frac{1}{2} u^2 \mathrm{d}t$$

subject to the dynamic constraints

$$\dot{y} = v, \dot{v} = a, \dot{a} = u$$

where *y, v, a* and *u* are position, velocity, acceleration and jerk, respectively. Using the method of Lagrange multipliers, the constraints may be added to the scalar to be minimized as

$$msj = \frac{1}{t_2 - t_1} \times \int_{t_1}^{t_2} \left\{ \frac{1}{2} u^2 + \lambda_y (v - \dot{y}) + \lambda_v (a - \dot{v}) + \lambda_a (u - \dot{a}) \right\} \mathrm{d}t$$

where λ_y, λ_v and λ_a are functions to be determined. For convenience, form the Hamiltonian

$$H = \frac{1}{2}u^2 + \lambda_y v + \lambda_v a + \lambda_a u$$

so that

$$msj = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \left(H - \lambda_y \dot{y} - \lambda_v \dot{v} - \lambda_a \dot{a}\right)\mathrm{d}t.$$

For simplicity, assume a time scale such that $t_1 = 0$ and $t_2 = 1$. Application of variational calculus shows that within the interval $0 \le t \le 1$ the partial derivative of H with respect to each of its arguments must be zero. The dynamic constraint equations (or state equations) are recovered from

$$\partial H/\partial\lambda_y\big|_o = v = \dot{y}, \partial H/\partial\lambda_v\big|_o = a = \dot{v} \text{ and } \partial H/\partial\lambda_a\big|_o = u = \dot{a}.$$

The Lagrange multipliers are determined by the co-state equations

$$\partial H/\partial y\big|_o = 0 = -\dot{\lambda}_y, \partial H/\partial v\big|_o = \lambda_y = -\dot{\lambda}_v$$
$$\text{and } \partial H/\partial a\big|_o = \lambda_v = -\dot{\lambda}_a.$$

The optimal solution is defined by

$$\partial H/\partial u\big|_o = 0 = u^o + \lambda_a.$$

Combining equations, $-\lambda_a$ is jerk, λ_v is snap (the fourth time derivative of position), $-\lambda_y$ is crackle (the fifth time derivative of position) and the minimal mean-squared jerk movement is defined by

$$\mathrm{d}^6 y/\mathrm{d}t^6 = 0.$$

Integrating yields a fifth-order polynomial

$$y^0(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

with six coefficients to be determined by the boundary conditions at the ends of the time interval. A discrete movement starting from rest at $y(0) = 0$ and ending at rest at $y(1) = 1$ yields

$$a_0 = a_1 = a_2 = 0,\ a_3 = 10,\ a_4 = -15 \text{ and } a_5 = 6$$

so that the general solution is

$$y^0(t) = y(0) + A\left[10\left(\frac{t}{d}\right)^3 - 15\left(\frac{t}{d}\right)^4 + 6\left(\frac{t}{d}\right)^5\right]$$

for $0 \le t \le 1$ where $y(0)$ is initial position, A is movement amplitude and d is movement duration. The minimal value of the *msj* measure for this movement is

$$msj^0 = 360A^2/d^6.$$

To identify the smoothest cyclic movement between two positions, consider two adjacent intervals, i.e., a ‘‘forth’’ movement from the first to the second position in the interval $0 \le t \le m$, and a ‘‘back’’ movement from the second to the first position in the interval $m \le t \le p$, where p is the period of the cycle and $0 < m < p$ is the *passage time* (to be determined) at which the second position, y_2, is passed. The details of movement between these positions may be found by solving an optimization problem with an ‘‘interior-point constraint’’ N on position at the passage time m such that

$$N = y(m) - y_2 = 0.$$

Again using the method of Lagrange multipliers, this constraint may be appended to the measure to be minimized to form

$$msj_c = \pi N + \frac{1}{p}\int_0^p \frac{1}{2}u^2\mathrm{d}t.$$

The integral has two components, the first with a variable end time, the second with a variable start time,

$$msj_c = \pi N + \frac{1}{p}\left[\int_0^m \frac{1}{2}u^2\mathrm{d}t + \int_m^p \frac{1}{2}u^2\mathrm{d}t\right].$$

Applying variational calculus, within each interval the movement is described by a quintic polynomial as above. Thirteen coefficients (six for each interval plus the passage time, m) have to be determined from the boundary conditions at the end of each interval. For simplicity, assume $p = 2$. The position, velocity and acceleration (indeed, all derivatives) at the beginning of the ‘‘forth’’ interval and end of the ‘‘back’’ interval are identical:

$$y(2) = y(0) = 0,\ v(2) = v(0),\ a(2) = a(0)$$

and so on; this is a basic requirement for a cyclic movement. At the passage time, position is known but none of its time derivatives are. In general, the Hamiltonian and each of the Lagrange multipliers λ_y, λ_v and λ_a may be discontinuous at time m. Using self-evident subscripts for the first and second intervals, a general boundary condition at time m is

$$\mathrm{d}y\left(-\lambda_{y1}+\pi+\lambda_{y2}\right)+\mathrm{d}v(-\lambda_{v1}+\lambda_{v2})$$
$$+\mathrm{d}a(-\lambda_{a1}+\lambda_{a2})+\mathrm{d}m(H_1-H_2)=0.$$

As each of the infinitesimal differentials dy, dv, da and dm are unspecified, this reduces to

$$H_1=H_2,\ \lambda_{a_1}=\lambda_{a_2},\ \lambda_{v_1}=\lambda_{v_2},\ \text{and}\ \lambda_{y_1}=\pi+\lambda_{y_2}.$$

Thus crackle may be discontinuous at the passage time but the Hamiltonian, jerk and snap are continuous and (by integration) acceleration, velocity and position are also continuous. If crackle is discontinuous and the Hamiltonian is continuous then velocity must be zero at the passage time.

While this analysis yields sufficient equations to determine the unknown coefficients, evaluating them requires solving a large number of simultaneous, nonlinear, algebraic equations. A simpler approach is to assume that the ''forth'' and ''back'' movements have the same (unknown) shape, though perhaps different durations. If so, the jerk measure for each interval is proportional to the square of movement amplitude and inversely proportional to the sixth power of interval duration so that

$$msj_c=\frac{1}{p}\left[\frac{CA^2}{m^5}+\frac{CA^2}{(p-m)^5}\right],$$

where C is a constant that depends on the details of the shape. Setting $p = 2$ and minimizing with respect to the passage time, m, yields a sixth-order polynomial with only one real-valued root at $m = 1$. Thus the smoothest cyclic movement has equal-duration ''forth'' and ''back'' segments, each a mirror-image of the other. Boundary conditions for the ''forth'' movement are

$$y(0)=0,\ y(1)=1,\ v(0)=-v(1),$$
$$a(0)=-a(1),\ u(0)=-u(1)\ \text{and}\ s(0)=-s(1),$$

where s is snap. They yield

$$a_0=a_1=0,\ a_2=5/2,\ a_3=0,\ a_4=-5/2\ \text{and}\ a_5=1$$

so that the general solution is

$$y^0(t)=y(0)+A\left[\frac{5}{2}\left(\left(\frac{t}{d}\right)^2-\left(\frac{t}{d}\right)^4\right)+\left(\frac{t}{d}\right)^5\right]$$

for $0 \leq t \leq d$ where $y(0)$ is initial position, A is movement amplitude and d is the duration of the outbound movement, so that $p = 2d$. The minimal value of the msj measure for this movement is

$$msj^0=60A^2/d^6.$$

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